Solutions to the Exercises in Elementary Differential Geometry

Chapter 1

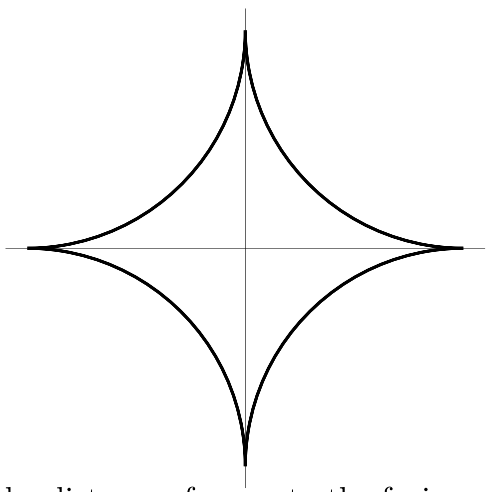
1.1.1 It is a parametrization of the part of the parabola with $x \geq 0$.

1.1.2 (i) $\boldsymbol{\gamma}(t) = (\sec t, \tan t)$ with $-\pi/2 < t < \pi/2$ and $\pi/2 < t < 3\pi/2$. Note that $\boldsymbol{\gamma}$ is defined on the union of two disjoint intervals: this corresponds to the fact that the hyperbola $y^2 - x^2 = 1$ is in two pieces, where $y \geq 1$ and where $y \leq -1$.
(ii) $\boldsymbol{\gamma}(t) = (2\cos t, 3\sin t)$.

1.1.3 (i) $x + y = 1$.
(ii) $y = (\ln x)^2$.

1.1.4 (i) $\dot{\boldsymbol{\gamma}}(t) = \sin 2t(-1, 1)$.
(ii) $\dot{\boldsymbol{\gamma}}(t) = (e^t, 2t)$.

1.1.5 $\dot{\boldsymbol{\gamma}}(t) = 3\sin t\cos t(-\cos t, \sin t)$ vanishes where $\sin t = 0$ or $\cos t = 0$, i.e. $t = n\pi/2$ where n is any integer. These points correspond to the four cusps of the astroid (see Exercise 1.3.3).



1.1.6 (i) The squares of the distances from $\mathbf{p}$ to the foci are

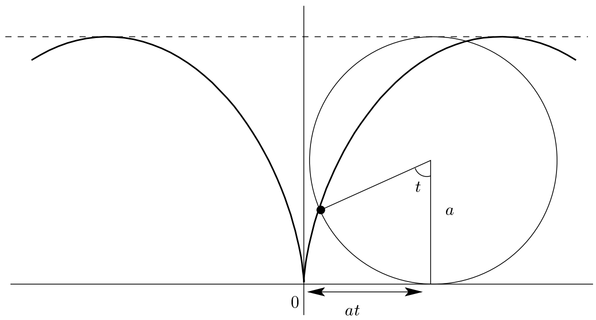
$$(p\cos t \pm \epsilon p)^2 + q^2 \sin^2 t = (p^2 - q^2)\cos^2 t \pm 2\epsilon p^2 \cos t + p^2 = p^2(1 \pm \epsilon \cos t)^2,$$

so the sum of the distances is $2p$.
(ii) $\dot{\boldsymbol{\gamma}} = (-p\sin t, q\cos t)$ so if $\mathbf{n} = (q\cos t, p\sin t)$ then $\mathbf{n}.\dot{\boldsymbol{\gamma}} = 0$. Hence the distances from the foci to the tangent line at $\boldsymbol{\gamma}(t)$ are

$$\frac{(p\cos t \mp \epsilon p, q\sin t).\mathbf{n}}{\| \, \mathbf{n} \, \|} = \frac{pq(1 \mp \epsilon \cos t)}{(p^2 \sin^2 t + q^2 \cos^2 t)^{1/2}}$$

and their product is $\frac{p^2q^2(1-\epsilon^2\cos^2 t)}{(p^2\sin^2 t+q^2\cos^2 t)} = q^2$.

(iii) It is enough to prove that $\frac{(\mathbf{p}-\mathbf{f}_1).\mathbf{n}}{\|\mathbf{p}-\mathbf{f}_1\|} = \frac{(\mathbf{p}-\mathbf{f}_2).\mathbf{n}}{\|\mathbf{p}-\mathbf{f}_2\|}$. Computation shows that both sides are equal to q.



1.1.7 When the circle has rotated through an angle t, its centre has moved to (at, a), so the point on the circle initially at the origin is now at the point

$$(a(t-\sin t), a(1-\cos t))$$

(see the diagram above).

1.1.8 Suppose that a point (x, y, z) lies on the cylinder if $x^2+y^2 = 1/4$ and on the sphere if $(x+\frac{1}{2})^2+y^2+z^2 = 1$. From the second equation, $-1 \le z \le 1$ so let $z = \sin t$. Subtracting the two equations gives $x + \frac{1}{4} + \sin^2 t = \frac{3}{4}$, so $x = \frac{1}{2} - \sin^2 t = \cos^2 t - \frac{1}{2}$. From either equation we then get $y = \sin t \cos t$ (or $y = -\sin t \cos t$, but the two solutions are interchanged by $t \mapsto \pi - t$).

1.1.9 $\dot{\boldsymbol{\gamma}} = (-2\sin t + 2\sin 2t, 2\cos t - 2\cos 2t) = \sqrt{2}(\sqrt{2}-1, 1)$ at $t = \pi/4$. So the tangent line is $y - (\frac{1}{\sqrt{2}} - 1) = (x-\sqrt{2})/(\sqrt{2}-1)$ and the normal line is

$$y - (\frac{1}{\sqrt{2}} - 1) = -(x-\sqrt{2})(\sqrt{2}-1).$$

1.1.10 (i) Putting $x = \sec t$ gives $y = \pm\sec t\tan t$ so $\boldsymbol{\gamma}(t) = (\sec t, \pm\sec t\tan t)$ gives parametrizations of the two pieces of this curve ($x \ge 1$ and $x \le -1$).

(ii) Putting $y = tx$ gives $x = \frac{3t}{1+t^3}$, $y = \frac{3t^2}{1+t^3}$.

1.1.11 (i) From $x = 1+\cos t$, $y = \sin t(1+\cos t)$ we get $y = x\sin t$ so

$$y^2 = x^2(1-(x-1)^2) = x^3(2-x).$$

(ii) $y = tx$ so $x^4 = t^2x^3 + t^3x^3 = y^2x + y^3 = y^2(x+y)$.

1.1.12 (i) $\dot{\boldsymbol{\gamma}}(t) = (-\sin t, \cos t + \cos 2t)$ so $\dot{\boldsymbol{\gamma}}(t) = \mathbf{0}$ if and only if $t = n\pi$ ($n \in \mathbb{Z}$) and $(-1)^n + 1 = 0$, so n must be odd.

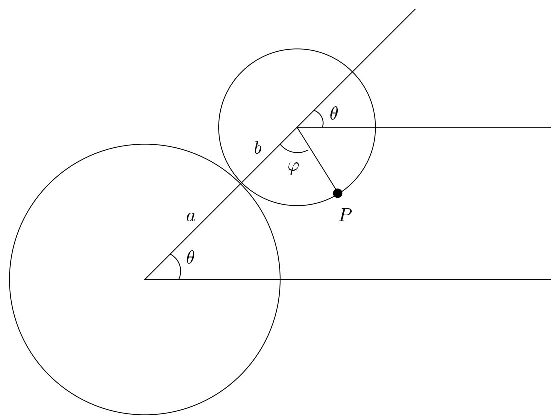
(ii) $\dot{\boldsymbol{\gamma}}(t) = (2t+3t^2, 3t^2+4t^3)$. This vanishes $\iff t(2+3t) = 0$ and $t^2(3+4t) = 0$, i.e. $\iff t = 0$.

1.1.13 (i) Let OP make an angle θ with the positive x-axis. Then R has coordinates $\boldsymbol{\gamma}(\theta) = (2a\cot\theta, a(1-\cos 2\theta))$.
(ii) From $x = 2a\cot\theta$, $y = a(1-\cos 2\theta)$, we get $\sin^2\theta = y/2a$, $\cos^2\theta = \cot^2\theta\sin^2\theta = x^2y/8a^3$, so the Cartesian equation is $y/2a + x^2y/8a^3 = 1$.

1.1.14 Let the fixed circle have radius a, and the moving circle radius b (so that $b < a$ in the case of the hypocycloid), and let the point P of the moving circle be initially in contact with the fixed circle at $(a, 0)$. When the moving circle has rotated through an angle φ, the line joining the origin to the point of contact of the circles makes an angle θ with the positive x-axis, where $a\theta = b\varphi$. The point P is then at the point

$$\begin{aligned}\boldsymbol{\gamma}(\theta) &= ((a+b)\cos\theta - b\cos(\theta+\varphi), (a+b)\sin\theta - b\sin(\theta+\varphi)) \\ &= \left((a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right), (a+b)\sin\left(\frac{a+b}{b}\theta\right)\right)\end{aligned}$$

in the case of the epicycloid,



and

$$\begin{aligned}\boldsymbol{\gamma}(\theta) &= ((a-b)\cos\theta + b\cos(\varphi-\theta), (a-b)\sin\theta - b\sin(\varphi-\theta)) \\ &= \left((a-b)\cos\theta - b\cos\left(\frac{a-b}{b}\theta\right), (a-b)\sin\left(\frac{a-b}{b}\theta\right)\right)\end{aligned}$$

in the case of the hypocycloid.

1.1.15 $\dot{\boldsymbol{\gamma}}(t) = (e^t(\cos t - \sin t), e^t(\sin t + \cos t))$ so the angle θ between $\boldsymbol{\gamma}(t)$ and $\dot{\boldsymbol{\gamma}}(t)$ is given by

$$\cos\theta = \frac{\boldsymbol{\gamma}.\dot{\boldsymbol{\gamma}}}{\parallel \boldsymbol{\gamma} \parallel \parallel \dot{\boldsymbol{\gamma}} \parallel} = \frac{e^{2t}(\cos^2 t - \sin t\cos t + \sin^2 t + \sin t\cos t)}{e^{2t}((\cos t - \sin t)^2 + (\sin t + \cos t)^2)} = \frac{1}{2},$$

so $\theta = \pi/3$.

1.1.16 $\dot{\boldsymbol{\gamma}}(t) = (t\cos t, t\sin t)$ so a unit tangent vector is $\mathbf{t} = (\cos t, \sin t)$. The distance of the normal line at $\boldsymbol{\gamma}(t)$ from the origin is

$$|\boldsymbol{\gamma}(t).\mathbf{t}| = |\cos^2 t + t\sin t\cos t + \sin^2 t - t\sin t\cos t| = 1.$$

1.2.1 $\dot{\boldsymbol{\gamma}}(t) = (1, \sinh t)$ so $\parallel \dot{\boldsymbol{\gamma}} \parallel = \cosh t$ and the arc-length is $s = \int_0^t \cosh u\, du = \sinh t$.

1.2.2 (i) $\parallel \dot{\boldsymbol{\gamma}} \parallel^2 = \frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2} = 1$.
(ii) $\parallel \dot{\boldsymbol{\gamma}} \parallel^2 = \frac{16}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t = \cos^2 t + \sin^2 t = 1$.

1.2.3 Denoting $d/d\theta$ by a dot, $\dot{\boldsymbol{\gamma}} = (\dot{r}\cos\theta - r\sin\theta, \dot{r}\sin\theta + r\cos\theta)$ so $\parallel \dot{\boldsymbol{\gamma}} \parallel^2 = \dot{r}^2 + r^2$. Hence, $\boldsymbol{\gamma}$ is regular unless $r = \dot{r} = 0$ for some value of θ. It is unit-speed if and only if $\dot{r}^2 = 1 - r^2$, which gives $r = \pm\sin(\theta + \alpha)$ for some constant α. To see that this is the equation of a circle of radius $1/2$, see the diagram in the proof of Theorem 3.2.2.

1.2.4 Since $\mathbf{u}$ is a unit vector, $|\dot{\boldsymbol{\gamma}}.\mathbf{u}| = \parallel \dot{\boldsymbol{\gamma}} \parallel \cos\theta$, where θ is the angle between $\dot{\boldsymbol{\gamma}}$ and $\mathbf{u}$, so $\dot{\boldsymbol{\gamma}}.\mathbf{u} \leq \parallel \dot{\boldsymbol{\gamma}} \parallel$. Then, $(\mathbf{q} - \mathbf{p}).\mathbf{u} = (\boldsymbol{\gamma}(b) - \boldsymbol{\gamma}(a)).\mathbf{u} = \int_a^b \dot{\boldsymbol{\gamma}}.\mathbf{u}\, dt \leq \int_a^b \parallel \dot{\boldsymbol{\gamma}} \parallel\, dt$. Taking $\mathbf{u} = (\mathbf{q} - \mathbf{p})/ \parallel \mathbf{q} - \mathbf{p} \parallel$ gives the result.

1.2.5 $\dot{\boldsymbol{\gamma}}(t) = (6t, 1 - 9t^2)$ so $\parallel \dot{\boldsymbol{\gamma}}(t) \parallel = \sqrt{36t^2 + 1 - 18t^2 + 81t^4} = 1 + 9t^2$. So the arc-length is

$$s = \int_0^t (1 + 9u^2) du = t + 3t^3.$$

1.2.6 We have $\dot{\boldsymbol{\gamma}}(t) = (-2\sin t + 2\sin 2t, 2\cos t - 2\cos 2t)$ so

$$\parallel \dot{\boldsymbol{\gamma}}(t) \parallel = \sqrt{8(1 - \sin t\sin 2t - \cos t\cos 2t} = \sqrt{8(1 - \cos t)} = 4\sin\frac{t}{2}.$$

So the arc-length is

$$s = \int_0^x 4\sin\frac{t}{2} dt = 8\left(1 - \cos\frac{x}{2}\right) = 16\sin^2\frac{x}{4}.$$

1.2.7 The cycloid is parametrized by $\boldsymbol{\gamma}(t) = a(t - \sin t, 1 - \cos t)$, where t is the angle through which the circle has rotated. So

$$\dot{\boldsymbol{\gamma}} = a(1 - \cos t, \sin t), \quad \parallel \dot{\boldsymbol{\gamma}} \parallel^2 = a^2(2 - 2\cos t) = 4a^2\sin^2\frac{t}{2},$$

and the arc-length is

$$\int_0^{2\pi} 2a\sin\frac{t}{2}\, dt = -4a\cos\frac{t}{2}\bigg|_{t=0}^{t=2\pi} = 8a.$$

1.2.8 The curve intersects the x-axis where $\cosh t = 3$, say at $t = \pm a$ where $a > 0$. We have $\| \dot{\boldsymbol{\gamma}}(t) \|^2 = 2\cosh^2 t - 2\cosh t$ so the arc-length is

$$s = \int_{-a}^{a} \sqrt{2\cosh^2 t - 2\cosh t}\, dt = 2\int_{0}^{a} \sqrt{2\cosh^2 t - 2\cosh t}\, dt.$$

To evaluate the integral put $u = \cosh t$. Then,

$$s = 2\int_1^3 \sqrt{\frac{u}{u+1}}du = \left[\sqrt{u(u+1)} - \ln(\sqrt{u} + \sqrt{u+1})\right]_1^3 = 2\sqrt{3} - \sqrt{2} - \ln\frac{2+\sqrt{3}}{1+\sqrt{2}}.$$

1.2.9 $\dddot{\boldsymbol{\gamma}} = \mathbf{0} \Longrightarrow \boldsymbol{\gamma} = \mathbf{a} + t\mathbf{b} + t^2\mathbf{c}$, where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are constant vectors. This implies that $\boldsymbol{\gamma}$ is contained in the plane passing through the point with position vector $\mathbf{a}$ and parallel to the vectors $\mathbf{b}$ and $\mathbf{c}$ (i.e. perpendicular to $\mathbf{b} \times \mathbf{c}$). (If $\mathbf{b}$ and $\mathbf{c}$ are parallel there are infinitely-many such planes.)

1.3.1 (i) $\dot{\boldsymbol{\gamma}} = \sin 2t(-1, 1)$ vanishes when t is an integer multiple of $\pi/2$, so $\boldsymbol{\gamma}$ is not regular.
(ii) $\boldsymbol{\gamma}$ is regular since $\dot{\boldsymbol{\gamma}} \neq \mathbf{0}$ for $0 < t < \pi/2$.
(iii) $\dot{\boldsymbol{\gamma}} = (1, \sinh t)$ is obviously never zero, so $\boldsymbol{\gamma}$ is regular.

1.3.2 $x = r\cos\theta = \sin^2\theta$, $y = r\sin\theta = \sin^2\theta\tan\theta$, so the parametrization in terms of θ is $\theta \mapsto (\sin^2\theta, \sin^2\theta\tan\theta)$. Since $\theta \mapsto \sin\theta$ is a bijective smooth map $(-\pi/2, \pi/2) \to (-1, 1)$, with smooth inverse $t \mapsto \sin^{-1} t$, $t = \sin\theta$ is a reparametrization map. Since $\sin^2\theta = t^2$, $\sin^2\theta\tan\theta = t^3/\sqrt{1-t^2}$, so the reparametrized curve is as stated.

1.3.3 (i) $\dot{\boldsymbol{\gamma}} = \mathbf{0}$ at $t = 0 \Longleftrightarrow m$ and n are both ≥ 2. If $m > 3$ the first components of $\ddot{\boldsymbol{\gamma}}$ and $\dddot{\boldsymbol{\gamma}}$ are both 0 at $t = 0$ so $\ddot{\boldsymbol{\gamma}}$ and $\dddot{\boldsymbol{\gamma}}$ are linearly dependent at $t = 0$; similarly if $n > 3$. So there are four cases: if $(m, n) = (2, 2)$ or $(3, 3)$ then either $\ddot{\boldsymbol{\gamma}}$ or $\dddot{\boldsymbol{\gamma}}$ is zero at $t = 0$, so the only possibilities for an ordinary cusp are $(m, n) = (2, 3)$ and $(3, 2)$ and then $\ddot{\boldsymbol{\gamma}}$ and $\dddot{\boldsymbol{\gamma}}$ are easily seen to be linearly independent at $t = 0$.
(ii) Using the parametrization $\boldsymbol{\gamma}(t) = \left(t^2, \frac{t^3}{\sqrt{1-t^2}}\right)$, we get $\dot{\boldsymbol{\gamma}} = \mathbf{0}$, $\ddot{\boldsymbol{\gamma}} = (2, 0)$, $\dddot{\boldsymbol{\gamma}} = (0, 6)$ at $t = 0$ so the origin is an ordinary cusp.
(iii) Let $\tilde{\boldsymbol{\gamma}}(\tilde{t})$ be a reparametrization of $\boldsymbol{\gamma}(t)$, and suppose $\boldsymbol{\gamma}$ has an ordinary cusp at $t = t_0$. Then, at $t = t_0$, $d\tilde{\boldsymbol{\gamma}}/d\tilde{t} = (d\boldsymbol{\gamma}/dt)(dt/d\tilde{t}) = 0$, $d^2\tilde{\boldsymbol{\gamma}}/d\tilde{t}^2 = (d^2\boldsymbol{\gamma}/dt^2)(dt/d\tilde{t})^2$, $d^3\tilde{\boldsymbol{\gamma}}/d\tilde{t}^3 = (d^3\boldsymbol{\gamma}/dt^3)(dt/d\tilde{t})^3 + 3(d^2\boldsymbol{\gamma}/dt^2)(dt/d\tilde{t})(d^2t/d\tilde{t}^2)$. Using the fact that $dt/d\tilde{t} \neq 0$, it is easy to see that $d^2\tilde{\boldsymbol{\gamma}}/d\tilde{t}^2$ and $d^3\tilde{\boldsymbol{\gamma}}/d\tilde{t}^3$ are linearly independent when $t = t_0$.

1.3.4 (i) If $\tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(\varphi(t))$, let ψ be the inverse of the reparametrization map φ. Then $\tilde{\boldsymbol{\gamma}}(\psi(t)) = \boldsymbol{\gamma}(\varphi(\psi(t))) = \boldsymbol{\gamma}(t)$.
(ii) If $\tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(\varphi(t))$ and $\hat{\boldsymbol{\gamma}}(t) = \tilde{\boldsymbol{\gamma}}(\psi(t))$, where φ and ψ are reparametrization maps, then $\hat{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}((\varphi \circ \psi)(t))$ and $\varphi \circ \psi$ is a reparametrization map because it is smooth and $\frac{d}{dt}(\varphi(\psi(t)) = \dot{\varphi}(\psi(t))\dot{\psi}(t) \neq 0$ as $\dot{\varphi}$ and $\dot{\psi}$ are both $\neq 0$.

1.3.5 (i) $\dot{\boldsymbol{\gamma}}(t) = (2t, 3t^3)$ which vanishes $\iff t = 0$. So $\boldsymbol{\gamma}$ is not regular.
(ii) $\dot{\boldsymbol{\gamma}}(t) = (-\sin t - \sin 2t, \cos t + \cos 2t)$ so $\| \dot{\boldsymbol{\gamma}}(t) \| = 2\cos\frac{t}{2}$. Hence, $\dot{\boldsymbol{\gamma}}(t) = \mathbf{0} \iff t$ is an odd multiple of π. Thus, when t is retricted to the interval $-\pi < t < \pi$, $\boldsymbol{\gamma}$ is regular.

1.3.6 $\dot{\boldsymbol{\gamma}}(t) = \left(2, -\frac{4t}{(1+t^2)^2}\right)$ is obviously never zero, so $\boldsymbol{\gamma}$ is regular. Let $\varphi(t) = \frac{\cos t}{1+\sin t}$. Then, $\frac{d\varphi}{dt} = -\frac{1}{1+\sin t}$ is never zero so φ is a reparametrization map. Setting $u = \varphi(t)$ we get

$$\boldsymbol{\gamma}(u) = \left(\frac{2\cos t}{1+\sin t}, \frac{2}{1+\frac{\cos^2 t}{(1+\sin t)^2}}\right) = \left(\frac{2\cos t}{1+\sin t}, \frac{2(1+\sin t)^2}{2+2\sin t}\right) = \tilde{\boldsymbol{\gamma}}(t),$$

so $\boldsymbol{\gamma}$ is a reparametrization of $\tilde{\boldsymbol{\gamma}}$.

1.3.7 $\dot{\boldsymbol{\gamma}}(t) = (a\omega\cos\omega t, b\cos t) = \mathbf{0} \iff \cos\omega t = \cos t = 0 \iff \omega t = (2k+1)\frac{\pi}{2}$ and $t = (2l+1)\frac{\pi}{2}$, where $k, l \in \mathbb{Z}$. This means that $\omega = \frac{2k+1}{2l+1}$ is the ratio of two odd integers.

1.3.8 We have $s = \int_{t_0}^{t} \| d\boldsymbol{\gamma}/du \| \; du$, $\tilde{s} = \int_{\tilde{t}_0}^{\tilde{t}} \| d\tilde{\boldsymbol{\gamma}}/d\tilde{u} \| \; d\tilde{u}$. By the chain rule, $d\boldsymbol{\gamma}/du = (d\tilde{\boldsymbol{\gamma}}/d\tilde{u})(d\tilde{u}/du)$, so $s = \pm\int_{t_0}^{t} \| d\tilde{\boldsymbol{\gamma}}/d\tilde{u} \| \; (d\tilde{u}/du)\, du = \pm\tilde{s}$, the sign being that of $d\tilde{u}/du$.

1.3.9 We can assume that the curve $\boldsymbol{\gamma}$ is unit-speed and that the tangent lines all pass through the origin (by applying a translation to $\boldsymbol{\gamma}$). Then, there is a scalar $\lambda(t)$ such that $\dot{\boldsymbol{\gamma}}(t) = \lambda(t)\boldsymbol{\gamma}(t)$ for all t. Then, $\ddot{\boldsymbol{\gamma}} = \dot{\lambda}\boldsymbol{\gamma} + \lambda\dot{\boldsymbol{\gamma}} = (\dot{\lambda} + \lambda^2)\boldsymbol{\gamma}$. This implies that $\ddot{\boldsymbol{\gamma}}$ is parallel to $\dot{\boldsymbol{\gamma}}$. But these vectors are perpendicular since $\boldsymbol{\gamma}$ is unit-speed. Hence, $\ddot{\boldsymbol{\gamma}} = \mathbf{0}$, so $\boldsymbol{\gamma}(t) = t\mathbf{a} + \mathbf{b}$ where $\mathbf{a}, \mathbf{b}$ are constant vectors, i.e. $\boldsymbol{\gamma}$ is a straight line.

If all the normal lines are parallel, so are all the tangent lines. If $\boldsymbol{\gamma}$ is assumed to be unit-speed, this means that $\dot{\boldsymbol{\gamma}}$ is constant, say equal to $\mathbf{a}$. Then $\boldsymbol{\gamma}(t) = t\mathbf{a} + \mathbf{b}$ as before.

1.4.1 It is closed because $\boldsymbol{\gamma}(t+2\pi) = \boldsymbol{\gamma}(t)$ for all t. Suppose that $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(u)$. Then $\cos^3 t(\cos 3t, \sin 3t) = \cos^3 u(\cos 3u, \sin 3u)$. Taking lengths gives $\cos^3 t = \pm\cos^3 u$ so $\cos t = \pm\cos u$, so $u = t, \pi - t, \pi + t$ or $2\pi - t$ (up to adding multiples of 2π). The second possibility forces $t = n\pi/3$ for some integer n and the third possibility is true for all t. Hence, the period is π and for the self-intersections we need only consider $t = \pi/3, 2\pi/3$, giving $u = 2\pi/3, \pi/3$, respectively. Hence, there is a unique self-intersection at $\boldsymbol{\gamma}(\pi/3) = (-1/8, 0)$.

1.4.2 The curve $\tilde{\boldsymbol{\gamma}}(t) = (\cos(t^3+t), \sin(t^3+t))$ is a reparametrization of the circle $\boldsymbol{\gamma}(t) = (\cos t, \sin t)$ but it is not closed.

1.4.3 If $\boldsymbol{\gamma}$ is T-periodic then it is kT-periodic for all $k \neq 0$ (this can be proved by induction on k if $k > 0$, or on $-k$ if $k < 0$). If $\boldsymbol{\gamma}$ is T_1-periodic and T_2-periodic

then it is k_1T_1- and k_2T_2-periodic for all non-zero integers k_1, k_2, so

$$\boldsymbol{\gamma}(t + k_1T_1 + k_2T_2) = \boldsymbol{\gamma}(t + k_1T_1)$$

as $\boldsymbol{\gamma}$ is k_2T_2-periodic, which $= \boldsymbol{\gamma}(t)$ as $\boldsymbol{\gamma}$ is k_1T_1-periodic.

1.4.4 If $\boldsymbol{\gamma}$ is T-periodic write $T = kT_0 + T_1$ where k is an integer and $0 \leq T_1 < T_0$. By Exercise 1.4.3 $\boldsymbol{\gamma}$ is T_1-periodic; if $T_1 > 0$ this contradicts the definition of T_0.

1.4.5 (i) Choose $T_1 > 0$ such that $\boldsymbol{\gamma}$ is T_1-periodic; then T_1 is not the smallest positive number with this property, so there is a $T_2 > 0$ such that $\boldsymbol{\gamma}$ is T_2-periodic. Iterating this argument gives the desired sequence.

(ii) The sequence $\{T_r\}_{r\geq 1}$ in (i) is decreasing and bounded below, so must converge to some $T_\infty \geq 0$. Then $\boldsymbol{\gamma}$ is T_∞-periodic because (using continuity of $\boldsymbol{\gamma}$) $\boldsymbol{\gamma}(t + T_\infty) = \lim_{r\to\infty} \boldsymbol{\gamma}(t + T_r) = \lim_{r\to\infty} \boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(t)$. By Exercise 1.4.3, $\boldsymbol{\gamma}$ is $(T_r - T_\infty)$-periodic for all $r \geq 1$, and this sequence of positive numbers converges to 0.

(iii) If $\{T_r\}$ is as in (i) and $T_r \to 0$ as $r \to \infty$, then by the mean value theorem $0 = (f(t+T_r) - f(t))/T_r = \dot{f}(t + \lambda T_r)$ for some $0 < \lambda < 1$. Letting $r \to \infty$ gives $\dot{f}(t) = 0$ for all t, so f is constant.

1.4.6 Following the hint, since $T_0 = (k_i/k)T_i$ is an integer multiple of T_i, each γ_i is T_0-periodic. Let $\mathcal{T}$ be the union of the finite sets of real numbers $\{T_i, 2T_i, \ldots, k_iT_i\}$ over all i such that γ_i is not constant, and let $\mathcal{P} = \{T' \in \mathcal{T} \mid \boldsymbol{\gamma} \text{ is } T'\text{-periodic}\}$. Then $\mathcal{P}$ is finite (because $\mathcal{T}$ is) and non-empty (because $T \in \mathcal{P}$). The smallest element of $\mathcal{P}$ is the smallest positive number T_0' such that $\boldsymbol{\gamma}$ is T_0'-periodic (since if $\boldsymbol{\gamma}$ is T'-periodic either $T' > T$ or $T' \in \mathcal{P}$). By Exercise 1.4.4, $T_0 = k'T_0'$ for some integer k' and then there are integers k_i' such that $T_0' = k_i'T_i$ for all i such that γ_i is not constant. Then, $k_iT_i/k = k'k_iT_i$ so kk' divides each k_i. As k is the largest such divisor, $k' = 1$, so $T_0 = T_0'$.

1.4.7 If $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(u)$ then $\frac{t^2-3}{t^2+1} = \frac{u^2-3}{u^2+1}$. This implies $t^2 = u^2$. This shows that $\boldsymbol{\gamma}$ is not periodic and that for a self-intersection we must have $t = -u$. The equation $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(-t)$ implies $\frac{t(t^2-3)}{t^2+1} = 0$, so $t = 0$ or $\pm\sqrt{3}$. Hence, the unique self-intersection is at $\boldsymbol{\gamma}(\sqrt{3}) = \boldsymbol{\gamma}(-\sqrt{3}) = \mathbf{0}$.

1.4.8 If $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(u)$ then $\sin t = \sin u$. Also, $\| \boldsymbol{\gamma}(t) \|^2 = \| \boldsymbol{\gamma}(u) \|^2$, which gives $5 + 4\cos t = 5 + 4\cos u$, hence $\cos t = \cos u$. So $t = u + 2n\pi$ for some integer n. We must have $\cos\omega(t+2n\pi) = \cos\omega t$ so $\omega(t+2n\pi) = \omega t + 2m\pi$ for some integer m. Hence, $\omega = m/n$ is rational. The period T of $\boldsymbol{\gamma}$ is the smallest positive number such that $\boldsymbol{\gamma}(t + T) = \boldsymbol{\gamma}(t)$ for all t. From the first part we know that $T = 2N\pi$ for some integer N. Then, N is the smallest positive integer N such that $\omega(t+2N\pi) = \omega t + 2M\pi$ for some integer M, i.e. the smallest positive integer such that mN/n is an integer. If m and n have no common factor, $N = n$.

1.5.1 $x(1-x^2) \geq 0 \Longleftrightarrow x \leq -1$ or $0 \leq x \leq 1$ so the curve is in (at least) two pieces. The parametrization is defined for $t \leq -1$ and $0 \leq t \leq 1$ and it covers the part of the curve with $y \geq 0$.

1.5.2 If $\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$ is a curve in the surface $f(x, y, z) = 0$, differentiating $f(x(t), y(t), z(t)) = 0$ with respect to t gives $\dot{x}f_x + \dot{y}f_y + \dot{z}f_z = 0$, so $\dot{\boldsymbol{\gamma}}$ is perpendicular to $\boldsymbol{\nabla} f = (f_x, f_y, f_z)$. Since this holds for every curve in the surface, $\boldsymbol{\nabla} f$ is perpendicular to the surface. The surfaces $f = 0$ and $g = 0$ should intersect in a curve if the vectors $\boldsymbol{\nabla} f$ and $\boldsymbol{\nabla} g$ are not parallel at any point of the intersection.

1.5.3 Let $\boldsymbol{\gamma}(t) = (u(t), v(t), w(t))$ be a regular curve in $\mathbb{R}^3$. At least one of $\dot{u}, \dot{v}, \dot{w}$ is non-zero at each value of t. Suppose that $\dot{u}(t_0) \neq 0$ and $x_0 = u(t_0)$. As in the 'proof' of Theorem 1.5.2, there is a smooth function $h(x)$ defined for x near x_0 such that $t = h(x)$ is the unique solution of $x = u(t)$ for each t near t_0. Then, for t near t_0, $\boldsymbol{\gamma}(t)$ is contained in the level curve $f(x, y, z) = g(x, y, z) = 0$, where $f(x, y, z) = y - v(h(x))$ and $g(x, y, z) = z - w(h(x))$. The functions f and g satisfy the conditions in the previous exercise, since $\boldsymbol{\nabla} f = (-\dot{v}h', 1, 0)$, $\boldsymbol{\nabla} g = (-\dot{w}h', 0, 1)$, a dash denoting d/dx.

1.5.4 The equation cannot be satisfied if $x = 1$, so the line $x = 1$ separates the curve into at least two connected pieces. The curve $\boldsymbol{\gamma}$ is a parametrization because

$$\begin{aligned}(x-1)^2(x^2+y^2) &= \cos^2 t((1+\cos t)^2 + (\sin t + \tan t)^2) \\ &= \cos^2 t(1+\cos t)^2(1+\tan^2 t) = (1+\cos t)^2 = x^2.\end{aligned}$$

But since $\tan t$ is undefined when $t = \pm\pi/2, \pm 3\pi/2, ...$, $\boldsymbol{\gamma}$ must be defined on one of the intervals $(-\pi/2, \pi/2)$, $(\pi/2, 3\pi/2)$, etc. The restriction of $\boldsymbol{\gamma}$ to the first of these intervals parametrizes the part of the curve with $x > 1$, the second gives the part with $x < 1$. This shows that the two parts of the curve with $x < 1$ and with $x > 1$ are connected.

1.5.5 Letting $f(x, y, z) = x^2 + y^2 = 1/4$, $g(x, y, z) = x^2 + y^2 + z^2 + x - 3/4$, we find that $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (2x, 2y, 0)$, $(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}) = (2x+1, 2y, 2z)$. If these vectors are parallel, $z = 0$; if $y \neq 0$ the vectors must then be equal and this is impossible; so $y = 0$ and then we must have $x = 1/2$ to satisfy both equations. Hence, the vectors are parallel only at the point $(1/2, 0, 0)$.

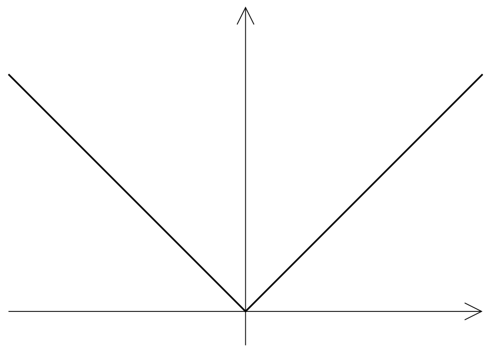
For the parametrization $\boldsymbol{\gamma}$ in Exercise 1.1.8, $\dot{\boldsymbol{\gamma}}(t) = (-\sin 2t, \cos 2t, \cos t)$. This is never zero as $\| \dot{\boldsymbol{\gamma}}(t) \|^2 = 1 + \cos^2 t$, so $\boldsymbol{\gamma}$ is regular.

1.5.6 Define $\Theta(t) = \tan \pi\theta(t)/2$, where θ is the function defined in Exercise 9.4.3. Then Θ is smooth, $\Theta(t) = 0$ if $t \leq 0$, and $\Theta : (0, \infty) \to (0, \infty)$ is a bijection. The curve

$$\boldsymbol{\gamma}(t) = \begin{cases} (\Theta(t), \Theta(t)) & \text{if } t \geq 0, \\ (-\Theta(-t), \Theta(-t)) & \text{if } t \leq 0, \end{cases}$$

is a smooth parametrization of $y = |x|$.

There is no regular parametrization of $y = |x|$. For if there were, there would be a unit-speed parametrization $\tilde{\boldsymbol{\gamma}}(t)$, say, and we can assume that $\tilde{\boldsymbol{\gamma}}(0) = (0,0)$. The unit tangent vector $\dot{\tilde{\boldsymbol{\gamma}}}$ would have to be either $\frac{1}{\sqrt{2}}(1,1)$ or $-\frac{1}{\sqrt{2}}(1,1)$ when $x > 0$, so by continuity we would have $\dot{\tilde{\boldsymbol{\gamma}}}(0) = \pm\frac{1}{\sqrt{2}}(1,1)$. But, by considering the part $x < 0$ in the same way, we see that $\dot{\tilde{\boldsymbol{\gamma}}}(0) = \pm\frac{1}{\sqrt{2}}(1,-1)$. These statements are contradictory.



Chapter 2

2.1.1 (i) $\boldsymbol{\gamma}$ is unit-speed (Exercise 1.2.2(i)) so

$$\kappa = \parallel \ddot{\boldsymbol{\gamma}} \parallel = \parallel (\frac{1}{4}(1+t)^{-1/2}, \frac{1}{4}(1-t)^{-1/2}, 0) \parallel = \frac{1}{\sqrt{8(1-t^2)}}.$$

(ii) $\boldsymbol{\gamma}$ is unit-speed (Exercise 1.2.2(ii)) so $\kappa = \parallel \ddot{\boldsymbol{\gamma}} \parallel = \parallel (-\frac{4}{5}\cos t, \sin t, \frac{3}{5}\cos t) \parallel = 1$.

(iii) $\kappa = \frac{\|(1,\sinh t,0)\times(0,\cosh t,0)\|}{\|(1,\sinh t,0)\|^3} = \frac{\cosh t}{\cosh^3 t} = \mathrm{sech}^2 t$ using Proposition 2.1.2.

(iv) $(-3\cos^2 t\sin t, 3\sin^2 t\cos t, 0)\times(-3\cos^3 t + 6\cos t\sin^2 t, 6\sin t\cos^2 t - 3\sin^3 t, 0) = (0,0,-9\sin^2 t\cos^2 t)$, so $\kappa = \frac{\|(0,0,-9\sin^2 t\cos^2 t)\|}{\|(-3\cos^2 t\sin t, 3\sin^2 t\cos t, 0)\|^3} = \frac{1}{3|\sin t\cos t|}$. This becomes infinite when t is an integer multiple of $\pi/2$, i.e. at the four cusps $(\pm1,0)$ and $(0,\pm1)$ of the astroid.

2.1.2 The proof of Proposition 1.3.5 shows that, if $\mathbf{v}(t)$ is a smooth (vector) function of t, then $\parallel \mathbf{v}(t) \parallel$ is a smooth (scalar) function of t provided $\mathbf{v}(t)$ is non-zero for all t. The result now follows from the formula in Proposition 2.1.2. The curvature of the regular curve $\boldsymbol{\gamma}(t) = (t, t^3)$ is $\kappa(t) = 6|t|/(1+9t^4)^{3/2}$, which is not differentiable at $t = 0$.

2.1.3 Working in $\mathbb{R}^3$, $\dot{\boldsymbol{\gamma}}(t) = (1-\cosh 2t, 2\sinh t, 0) = 2\sinh t(-\sinh t, 1, 0)$, $\ddot{\boldsymbol{\gamma}}(t) = (-2\sinh 2t, 2\cosh t, 0)$ which gives $\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}} = (0,0,4\sinh^2 t\cosh t)$ and $\kappa = \frac{\|\dot{\boldsymbol{\gamma}}\times\ddot{\boldsymbol{\gamma}}\|}{\|\dot{\boldsymbol{\gamma}}\|^3} = 1/2\sinh t\cosh^2 t$. This is non-zero if $t > 0$, but $\to 0$ as $t \to \infty$.

2.1.4 We have

$$\dot{\boldsymbol{\gamma}}(t) = (\sec t \tan t, \sec t \tan^2 t + \sec^3 t, 0),$$
$$\ddot{\boldsymbol{\gamma}}(t) = (2\sec^3 t - \sec t, 6\sec^3 t \tan t - \sec t \tan t),$$

which gives $\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} = \sec^4 t(0, 0, 2\sec^2 t - 3)$. Hence, the curvature vanishes where $\sec^2 t = 3/2$. If $-\pi/2 < t < \pi/2$, $\cos t > 0$ so the curvature vanishes at the two values of t between $-\pi/2$ and $\pi/2$ at which $\cos t = 2/3$, i.e. at the point $(3/2, \sqrt{5}/2)$.

2.2.1 Differentiate $\mathbf{t}.\mathbf{n}_s = 0$ and use $\dot{\mathbf{t}} = \kappa_s \mathbf{n}_s$.

2.2.2 If $\boldsymbol{\gamma}$ is smooth, $\mathbf{t} = \dot{\boldsymbol{\gamma}}$ is smooth and hence so are $\dot{\mathbf{t}}$ and $\mathbf{n}_s$ (since $\mathbf{n}_s$ is obtained by applying a rotation to $\mathbf{t}$). So $\kappa_s = \dot{\mathbf{t}}.\mathbf{n}_s$ is smooth.

2.2.3 For the first part, from the results in Appendix 1 it suffices to show that $\tilde{\kappa}_s = -\kappa_s$ if M is the reflection in a straight line l. But this is clear: if we take the fixed angle φ_0 in Proposition 2.2.1 to be the angle between l and the positive x-axis, then (in the obvious notation) $\tilde{\varphi} = -\varphi$. Conversely, if $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ have the same non-zero curvature, their signed curvatures are either the same or differ in sign. In the first case the curves differ by a direct isometry by Theorem 2.2.5; in the latter case, applying a reflection to one curve gives two curves with the same signed curvature, and these curves then differ by a direct isometry, so the original curves differ by an opposite isometry.

2.2.4 The first part is obvious as the effect of the dilation is to multiply s by a and leave φ unchanged.

For the second part, consider the small piece of the chain between the points with arc-length s and $s + \delta s$. The net horizontal force on this piece is (in the obvious notation) $\delta(T\cos\varphi)$, and as this must vanish $T\cos\varphi$ must be a constant, say λ. The net vertical force is $\delta(T\sin\varphi)$, and this must balance the weight of the piece of chain, which is a constant multiple of δs. This shows that $T\sin\varphi = \mu s + \nu$ for some constants μ, ν, and ν must be zero because $\varphi = s = 0$ at the lowest point of $\mathcal{C}$. From $T\cos\varphi = \lambda$, $T\sin\varphi = \mu s$, we get $\tan\varphi = s/a$ where $a = \lambda/\mu$. Hence, $\sec^2\varphi \frac{d\varphi}{ds} = 1/a$, so the signed curvature is

$$\kappa_s = d\varphi/ds = 1/a\sec^2\varphi = 1/a(1 + \tan^2\varphi) = \frac{a}{s^2 + a^2}.$$

Using the first part and Example 2.2.4 gives the result.

2.2.5 We have $d\boldsymbol{\gamma}^\lambda/dt = d\boldsymbol{\gamma}/dt + \lambda d\mathbf{n}_s/dt = (1 - \lambda\kappa_s)ds/dt\,\mathbf{t}$, so the arc-length s^λ of $\boldsymbol{\gamma}^\lambda$ satisfies $ds^\lambda/dt = |1 - \lambda\kappa_s|ds/dt$. The unit tangent vector of $\boldsymbol{\gamma}^\lambda$ is $\mathbf{t}^\lambda = (d\boldsymbol{\gamma}^\lambda/dt)/(ds^\lambda/dt) = \epsilon\mathbf{t}$, hence the signed unit normal of $\boldsymbol{\gamma}^\lambda$ is $\mathbf{n}_s^\lambda = \epsilon\mathbf{n}_s$. Then,

the signed curvature κ_s^λ of $\boldsymbol{\gamma}^\lambda$ is given by

$$\begin{aligned}\kappa_s^\lambda \mathbf{n}_s^\lambda &= \frac{d\mathbf{t}^\lambda}{ds^\lambda} = \frac{d\mathbf{t}^\lambda/dt}{|1-\lambda\kappa_s|(ds/dt)} = \frac{\epsilon}{|1-\lambda\kappa_s|}\frac{d\mathbf{t}}{ds} \\ &= \frac{\kappa_s}{1-\lambda\kappa_s}\mathbf{n}_s = \frac{\epsilon\kappa_s}{1-\lambda\kappa_s}\mathbf{n}_s^\lambda = \frac{\kappa_s}{|1-\lambda\kappa_s|}\mathbf{n}_s^\lambda.\end{aligned}$$

2.2.6 $\boldsymbol{\epsilon}(s_0)$ lies on the perpendicular bisector of the line joining $\boldsymbol{\gamma}(s_0)$ and $\boldsymbol{\gamma}(s_0+\delta s)$. This gives $(\boldsymbol{\epsilon}(s_0) - \frac{1}{2}(\boldsymbol{\gamma}(s_0)+\boldsymbol{\gamma}(s_0+\delta s))).(\boldsymbol{\gamma}(s_0+\delta s)-\boldsymbol{\gamma}(s_0)) = 0$. Using Taylor's theorem, and discarding terms involving powers of δs higher than the second, this gives (with all quantities evaluated at s_0) $(\boldsymbol{\epsilon}-\boldsymbol{\gamma}).\dot{\boldsymbol{\gamma}}\delta s + \frac{1}{2}(\boldsymbol{\epsilon}.\ddot{\boldsymbol{\gamma}} - 1 - \boldsymbol{\gamma}.\ddot{\boldsymbol{\gamma}})(\delta s)^2 = 0$. This must also hold when δs is replaced by $-\delta s$; adding and subtracting the two equations gives $(\boldsymbol{\epsilon}-\boldsymbol{\gamma}).\dot{\boldsymbol{\gamma}} = 0$ and $(\boldsymbol{\epsilon}-\boldsymbol{\gamma}).\ddot{\boldsymbol{\gamma}} = 1$. The first equation gives $\boldsymbol{\epsilon} = \boldsymbol{\gamma} + \lambda\mathbf{n}_s$ for some scalar λ, and since $\ddot{\boldsymbol{\gamma}} = \kappa_s\mathbf{n}_s$ the second gives $\lambda = 1/\kappa_s$.

2.2.7 The tangent vector of $\boldsymbol{\epsilon}$ is $\mathbf{t} + \frac{1}{\kappa_s}(-\kappa_s\mathbf{t}) - \frac{\dot\kappa_s}{\kappa_s^2}\mathbf{n}_s = -\frac{\dot\kappa_s}{\kappa_s^2}\mathbf{n}_s$ so its arc-length is $u = \int \parallel \dot{\boldsymbol{\epsilon}} \parallel\, ds = \int \frac{\dot\kappa_s}{\kappa_s^2} ds = u_0 - \frac{1}{\kappa_s}$, where u_0 is a constant. Hence, the unit tangent vector of $\boldsymbol{\epsilon}$ is $-\mathbf{n}_s$ and its signed unit normal is $\mathbf{t}$. Since $-d\mathbf{n}_s/du = \kappa_s\mathbf{t}/(du/ds) = \frac{\kappa_s^3}{\dot\kappa_s}\mathbf{t}$, the signed curvature of $\boldsymbol{\epsilon}$ is $\kappa_s^3/\dot\kappa_s$.

Any point on the normal line to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s)$ is $\boldsymbol{\gamma}(s) + \lambda\mathbf{n}_s(s)$ for some λ. Hence, the normal line intersects $\boldsymbol{\epsilon}$ at the point $\boldsymbol{\epsilon}(s)$, where $\lambda = 1/\kappa_s(s)$, and since the tangent vector of $\boldsymbol{\epsilon}$ there is parallel to $\mathbf{n}_s(s)$ by the first part, the normal line is tangent to $\boldsymbol{\epsilon}$ at $\boldsymbol{\epsilon}(s)$.

Denoting d/dt by a dash, $\boldsymbol{\gamma}' = a(1-\cos t, \sin t)$ so the arc-length s of $\boldsymbol{\gamma}$ is given by $ds/dt = 2a\sin(t/2)$ and $\mathbf{t} = d\boldsymbol{\gamma}/ds = (\sin(t/2), \cos(t/2))$. So $\mathbf{n}_s = (-\cos(t/2), \sin(t/2))$ and $\dot{\mathbf{t}} = (d\mathbf{t}/dt)/(ds/dt) = \frac{1}{4a\sin(t/2)}(\cos(t/2), -\sin(t/2)) = -1/4a\sin(t/2)\mathbf{n}_s$, so the signed curvature of $\boldsymbol{\gamma}$ is $-1/4a\sin(t/2)$ and its evolute

$$\begin{aligned}\boldsymbol{\epsilon}(t) &= a(t-\sin t, 1-\cos t) - 4a\sin(t/2)(-\cos(t/2), \sin(t/2)) \\ &= a(t+\sin t, -1+\cos t).\end{aligned}$$

Reparametrizing $\boldsymbol{\epsilon}$ by $\tilde t = \pi + t$, we get $a(\tilde t - \sin\tilde t, 1-\cos\tilde t) + a(-\pi, -2)$, so $\boldsymbol{\epsilon}$ is obtained from a reparametrization of $\boldsymbol{\gamma}$ by translating by the vector $a(-\pi, -2)$.

2.2.8 The free part of the string is tangent to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s)$ and has length $\ell - s$, hence the stated formula for $\boldsymbol{\iota}(s)$. The tangent vector of $\boldsymbol{\iota}$ is $\dot{\boldsymbol{\gamma}} - \dot{\boldsymbol{\gamma}} + (\ell-s)\ddot{\boldsymbol{\gamma}} = \kappa_s(\ell-s)\mathbf{n}_s$ (a dot denotes d/ds). The arc-length v of $\boldsymbol{\iota}$ is given by $dv/ds = \kappa_s(\ell-s)$ so its unit tangent vector is $\mathbf{n}_s$ and its signed unit normal is $-\mathbf{t}$. Now $d\mathbf{n}_s/dv = \frac{1}{\kappa_s(\ell-s)}\dot{\mathbf{n}}_s = \frac{-1}{\ell-s}\mathbf{t}$, so the signed curvature of $\boldsymbol{\iota}$ is $1/(\ell-s)$.

2.2.9 The arc-length parametrization of the catenary is $\tilde{\boldsymbol{\gamma}}(s) = (\sinh^{-1} s, \sqrt{1+s^2})$. The involute

$$\boldsymbol{\iota}(s) = \tilde{\boldsymbol{\gamma}}(s) - s\dot{\tilde{\boldsymbol{\gamma}}}(s) = \left(\sinh^{-1} s - \frac{s}{\sqrt{1+s^2}}, \frac{1}{\sqrt{1+s^2}}\right) = (u - \tanh u, \mathrm{sech} u)$$

if $u = \sinh^{-1} s$. Thus, if (x, y) is a point on the involute $\boldsymbol{\iota}$, $u = \cosh^{-1}(1/y)$ and $x = \cosh^{-1}(1/y) - \sqrt{1 - y^2}$.

2.2.10 The rotation $\rho_{-\theta(s)}$ takes the tangent line of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s)$ to l and the line joining q and $\boldsymbol{\gamma}(s)$ to a line parallel to that joining $\boldsymbol{\Gamma}(s)$ to $\mathbf{p} + s\mathbf{a}$. Hence,

$$\boldsymbol{\Gamma}(s) - (\mathbf{p} + s\mathbf{a}) = \rho_{-\theta(s)}(\mathbf{q} - \boldsymbol{\gamma}(s)),$$

which gives the stated equation. Now,

$$\dot{\boldsymbol{\Gamma}}(s) = \mathbf{a} + \left(\frac{d}{ds}\rho_{-\theta(s)}\right)(\mathbf{q} - \boldsymbol{\gamma}(s)) - \rho_{-\theta(s)}\dot{\boldsymbol{\gamma}}(s).$$

The last term is clearly parallel to $\mathbf{a}$ and as they are both unit vectors they are equal. So we want to prove that

$$\left(\frac{d}{ds}\rho_{-\theta(s)}\right)(\mathbf{q} - \boldsymbol{\gamma}(s)).\rho_{-\theta(s)}(\mathbf{q} - \boldsymbol{\gamma}(s)) = 0.$$

If $A = \rho_{-\theta(s)}$, $\mathbf{v} = \mathbf{q} - \boldsymbol{\gamma}(s)$, we have to show (in matrix notation) $(A\mathbf{v})^t \frac{dA}{ds}\mathbf{v} = 0$, i.e. $\mathbf{v}^t A^t \frac{dA}{ds}\mathbf{v} = 0$. Since A is orthogonal, this follows from parts (i) and (ii) of the hint.

To prove (i), use components: $\mathbf{v}^t S\mathbf{v} = \sum_{i,j} v_i v_j S_{ij} = \sum_{i,j} v_j v_i S_{ji} = -\sum_{i,j} v_i v_j S_{ij}$. For (ii), differentiate $A^t A = I$.

2.2.11 If two unit-speed curves have the same non-zero curvature, their signed curvatures are either the same or differ in sign. In the first case the curves differ by a direct isometry by Theorem 2.2.6; in the latter case, applying a reflection to one curve gives two curves with the same signed curvature by Exercise 2.2.3, and these curves then differ by a direct isometry.

2.2.12 By applying a translation we can assume that all the normal lines pass through the origin. Then, there is a scalar $\lambda(t)$ such that $\boldsymbol{\gamma}(t) = \lambda(t)\mathbf{n}_s(t)$ for all t. Differentiating gives $\mathbf{t} = \dot{\lambda}\mathbf{n}_s - \lambda\kappa_s\mathbf{t}$ using Exercise 2.2.1. It follows that $\dot{\lambda} = 0$, so λ is constant, and that $\lambda\kappa_s = -1$, so $\kappa_s = -1/\lambda$ is constant. Hence, $\boldsymbol{\gamma}$ is a circle by Example 2.2.7.

2.2.13 The arc-length

$$\begin{aligned} s &= \int \parallel \dot{\boldsymbol{\gamma}} \parallel \, dt = \int \parallel e^{kt}(k\cos t - \sin t, k\sin t + \cos t) \parallel dt \\ &= \int \sqrt{k^2 + 1}e^{kt}\, dt = \frac{\sqrt{k^2+1}}{k}e^{kt} + c, \end{aligned}$$

where c is a constant. Taking $c = 0$ makes $s \to 0$ as $t \to \mp\infty$ if $\pm k > 0$.

Since $\mathbf{t} = \dot{\boldsymbol{\gamma}}/ \parallel \dot{\boldsymbol{\gamma}} \parallel = \frac{1}{\sqrt{k^2+1}}(k\cos t - \sin t, k\sin t + \cos t)$, the signed unit normal is $\mathbf{n}_s = \frac{1}{\sqrt{k^2+1}}(-k\sin t - \cos t, k\cos t - \sin t)$. So

$$\frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}/dt}{ds/dt} = \frac{e^{-kt}}{k^2+1}(-k\sin t - \cos t, k\cos t - \sin t) = \frac{e^{-kt}}{\sqrt{k^2+1}}\mathbf{n}_s,$$

hence $\kappa_s = 1/ks$. By Theorem 2.2.6, any other curve with the same signed curvature is obtained from the logarithmic spiral by applying a direct isometry.

2.2.14 Let $\boldsymbol{\delta}(t)$ be the foot of the perpendicular from $\mathbf{p}$ to the tangent line of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t)$. Then $\boldsymbol{\delta} = \mathbf{p} + \lambda\mathbf{n}_s$ for some scalar $\lambda = (\boldsymbol{\delta} - \mathbf{p}).\mathbf{n}_s$. But $\boldsymbol{\gamma} - \boldsymbol{\delta}$ is parallel to the tangent line, so is perpendicular to $\mathbf{n}_s$: $(\boldsymbol{\gamma} - \boldsymbol{\delta}).\mathbf{n}_s = 0$. Hence, $\boldsymbol{\delta}.\mathbf{n}_s = \boldsymbol{\gamma}.\mathbf{n}_s$ and we get $\lambda = (\boldsymbol{\gamma} - \mathbf{p}).\mathbf{n}_s$.

Using Exercise 2.1.1, we find that $\dot{\boldsymbol{\delta}} = -\kappa_s(((\boldsymbol{\gamma}-\mathbf{p}).\mathbf{t})\mathbf{n}_s + ((\boldsymbol{\gamma}-\mathbf{p}).\mathbf{n}_s)\mathbf{t})$. So $\dot{\boldsymbol{\delta}} = \mathbf{0}$ if and only if either $\kappa_s = 0$ or $(((\boldsymbol{\gamma}-\mathbf{p}).\mathbf{t})\mathbf{n}_s + ((\boldsymbol{\gamma}-\mathbf{p}).\mathbf{n}_s)\mathbf{t}) = 0$. Since $\mathbf{n}_s$ and $\mathbf{t}$ are perpendicular, the last equation can hold only if $(\boldsymbol{\gamma}-\mathbf{p}).\mathbf{t} = (\boldsymbol{\gamma}-\mathbf{p}).\mathbf{n}_s = 0$, which implies that $\boldsymbol{\gamma} = \mathbf{p}$. Hence, $\boldsymbol{\delta}$ is regular if and only if $\boldsymbol{\gamma}$ has nowhere vanishing curvature and does not pass through $\mathbf{p}$.

For the circle $\boldsymbol{\gamma}(t) = (\cos t, \sin t)$, $\mathbf{n}_s = -\boldsymbol{\gamma}$ and so

$$\begin{aligned}\boldsymbol{\delta}(t) &= (-2,0) + (2+\cos t, \sin t).(\cos t, \sin t)(\cos t, \sin t)\\ &= ((1+2\cos t)\cos t - 2, (1+2\cos t)\sin t),\end{aligned}$$

which is obtained by translating the limaçon in Example 1.1.7 by $(-2,0)$.

2.2.15 (i) Differentiating $\boldsymbol{\gamma} = r\mathbf{t}$ gives $\mathbf{t} = \dot{r}\mathbf{t} + \kappa_s r\mathbf{n}_s$. Since $\mathbf{t}$ and $\mathbf{n}_s$ are perpendicular vectors, it follows that $\kappa_s = 0$ and $\boldsymbol{\gamma}$ is part of a straight line.

(ii) Differentiating $\boldsymbol{\gamma} = r\mathbf{n}_s$ gives $\mathbf{t} = \dot{r}\mathbf{n}_s + r\dot{\mathbf{n}}_s = \dot{r}\mathbf{n}_s - \kappa_s r\mathbf{t}$ (Exercise 2.1.1). Hence, $\dot{r} = 0$, so r is constant, and $\kappa_s = -1/r$, hence κ_s is constant. So $\boldsymbol{\gamma}$ is part of a circle.

(iii) Write $\boldsymbol{\gamma} = r(\mathbf{t}\cos\theta + \mathbf{n}_s\sin\theta)$. Differentiating and equating coefficients of $\mathbf{t}$ and $\mathbf{n}_s$ gives $\dot{r}\cos\theta - \kappa_s r\sin\theta = 1$, $\dot{r}\sin\theta + \kappa_s r\cos\theta = 0$, from which $\dot{r} = \cos\theta$ and $\kappa_s r = -\sin\theta$. From the first equation, $r = s\cos\theta$ (we can assume the arbitrary constant is zero by adding a suitable constant to s) so $\kappa_s = -1/s\cot\theta$. By Exercise 2.2.13, $\boldsymbol{\gamma}$ is obtained by applying a direct isometry to the logarithmic spiral defined there with $k = -\cot\theta$.

2.2.16 For the parabola $\boldsymbol{\gamma}(t) = (t, t^2)$, $\mathbf{t} = \frac{1}{\sqrt{1+4t^2}}(1, 2t)$, $\mathbf{n}_s = \frac{1}{\sqrt{1+4t^2}}(-2t, 1)$, so $\boldsymbol{\gamma}^\lambda(t) = \left(t - \frac{2\lambda}{\sqrt{1+4t^2}}, t^2 + \frac{\lambda}{\sqrt{1+4t^2}}\right)$. This gives $\dot{\boldsymbol{\gamma}}^\lambda = \left(1 - \frac{2\lambda}{(1+4t^2)^{3/2}}\right)(1, 2t)$. Hence, $\dot{\boldsymbol{\gamma}}^\lambda(t) = \mathbf{0} \iff 2\lambda = (1+4t^2)^{3/2}$. If $\lambda < 1/2$ this has no solution for t, so $\boldsymbol{\gamma}^\lambda$ is regular. If $\lambda = 1/2$, the only solution is $t = 0$ so $(0, 1/2)$ is the unique singular point of $\boldsymbol{\gamma}$. If $\lambda > 1/2$ there are two solutions for t, hence $\boldsymbol{\gamma}^\lambda$ has two singular points.

2.2.17 $\dot{F} = 2\dot{\boldsymbol{\gamma}}.(\boldsymbol{\gamma} - \mathbf{c})$. So $F(t_0) = \dot{F}(t_0) = 0 \iff \parallel \boldsymbol{\gamma}(t_0) - \mathbf{c} \parallel = R$ and $\dot{\boldsymbol{\gamma}}(t_0)$ is perpendicular to $\boldsymbol{\gamma}(t_0) - \mathbf{c}$. This means that $\boldsymbol{\gamma}(t_0)$ is on the circle $\mathcal{C}$ and that $\dot{\boldsymbol{\gamma}}(t_0) = \mathbf{t}(t_0)$ is parallel to the tangent to $\mathcal{C}$ at that point (since this tangent is perpendicular to the radius $\boldsymbol{\gamma}(t_0) - \mathbf{c}$).

Now $\ddot{F} = 2\dot{\mathbf{t}}.(\boldsymbol{\gamma} - \mathbf{c}) + 2$ (since $\boldsymbol{\gamma}$ is unit-speed), so $\ddot{F}(t_0) = 0 \iff \dot{\mathbf{t}}(t_0).(\boldsymbol{\gamma}(t_0) - \mathbf{c}) = -1$. Now $\dot{\mathbf{t}}(t_0) = \kappa_s(t_0)\mathbf{n}_s(t_0)$ is perpendicular to $\mathbf{t}(t_0)$ (since $\boldsymbol{\gamma}$ is unit-speed), hence parallel to $\boldsymbol{\gamma}(t_0) - \mathbf{c}$ from the condition $\dot{F}(t_0) = 0$. So the condition $\ddot{F}(t_0) = 0$ gives $\boldsymbol{\gamma}(t_0) - \mathbf{c} = \lambda \mathbf{n}_s(t_0)$ for some scalar λ and we have $\kappa_s(t_0)\mathbf{n}_s(t_0).\lambda\mathbf{n}_s(t_0) = -1$ which gives $\lambda = -1/\kappa_s(t_0)$. So the conditions $F(t_0) = \dot{F}(t_0) = \ddot{F}(t_0) = 0$ are satisfied if and only if the centre of $\mathcal{C}$ is $\mathbf{c} = \boldsymbol{\gamma}(t_0) + \frac{1}{\kappa_s(t_0)}\mathbf{n}_s(t_0)$ and its radius is $\parallel \boldsymbol{\gamma}(t_0) - \mathbf{c} \parallel = |\lambda| = 1/|\kappa_s(t_0)|$. Comparing with Exercise 2.2.6 shows that $\mathcal{C}$ is the osculating circle of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t_0)$.

2.2.18 From the solution of Exercise 2.2.16, $\mathbf{t} = \frac{1}{\sqrt{1+4t^2}}(1, 2t)$, $\mathbf{n}_s = \frac{1}{\sqrt{1+4t^2}}(-2t, 1)$, and if s is the arc-length of $\boldsymbol{\gamma}$, $ds/dt = \parallel \dot{\boldsymbol{\gamma}} \parallel = \sqrt{1+4t^2}$ (a dot denoting d/dt). Hence,

$$\frac{d\mathbf{t}}{ds} = \frac{1}{\sqrt{1+4t^2}}\dot{\mathbf{t}} = \left(-\frac{4t}{(1+4t^2)^2}, \frac{2}{(1+4t^2)^2}\right),$$

hence $\kappa_s = 2(1+4t^2)^{-3/2}$. The evolute is therefore

$$\boldsymbol{\epsilon} = \boldsymbol{\gamma} + \frac{1}{\kappa_s}\mathbf{n}_s = (t, t^2) + \frac{1}{2}(1+4t^2)(-2t, 1) = \left(-4t^3, 3t^2 + \frac{1}{2}\right).$$

2.2.19 If s is the arc-length of $\boldsymbol{\gamma}$, $ds/dt = \sqrt{a^2\sin^2 t + b^2\cos^2 t} = 1/\lambda$, say. Then, denoting d/ds by a dot, $\mathbf{t} = \dot{\boldsymbol{\gamma}} = \lambda(-a\sin t, b\cos t)$, $\mathbf{n}_s = \lambda(-b\cos t, -a\sin t)$,

$$\dot{\mathbf{t}} = \lambda[\lambda(-a\cos t, -b\sin t) - \lambda^3(a^2 - b^2)\sin t\cos t(-a\sin t, b\cos t)] = \lambda^3 ab\mathbf{n}_s$$

after some simplification. Hence, $\kappa_s = \lambda^3 ab = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}$ and so

$$\boldsymbol{\epsilon} = (a\cos t, b\sin t) + \frac{\lambda}{\lambda^3 ab}(-b\cos t, -a\sin t) = \left(\frac{a^2 - b^2}{a}\cos^3 t, \frac{b^2 - a^2}{b}\sin^3 t\right).$$

2.2.20 Let $\boldsymbol{\gamma}^\lambda = \boldsymbol{\gamma} + \lambda\mathbf{n}_s$. Then, $\dot{\boldsymbol{\gamma}}^\lambda = \dot{\boldsymbol{\gamma}} + \lambda\dot{\mathbf{n}}_s = (1 - \lambda\kappa_s)\mathbf{n}_s$. Assuming that $\lambda\kappa_s \neq 1$ (so that $\boldsymbol{\gamma}^\lambda$ is regular), we have (in an obvious notation) $\frac{ds^\lambda}{ds} = |1 - \lambda\kappa_s|$, $\mathbf{t}^\lambda = \pm\mathbf{t}$, $\mathbf{n}_s^\lambda = \pm\mathbf{n}_s$, where $\pm$ is the sign of $1 - \lambda\kappa_s$. Hence,

$$\frac{d\mathbf{t}^\lambda}{ds^\lambda} = \frac{\pm\dot{\mathbf{t}}}{|1 - \lambda\kappa_s|} = \pm\frac{\kappa_s\mathbf{n}_s}{|1 - \lambda\kappa_s|} = \frac{\kappa_s\mathbf{n}_s^\lambda}{|1 - \lambda\kappa_s|}.$$

So $\kappa_s^\lambda = \frac{\kappa_s}{|1-\lambda\kappa_s|}$ and

$$\boldsymbol{\epsilon}^\lambda = \boldsymbol{\gamma} + \lambda \mathbf{n}_s + \frac{|1-\lambda\kappa_s|}{\kappa_s}(\pm\mathbf{n}_s) = \boldsymbol{\gamma} + \lambda \mathbf{n}_s + \frac{1-\lambda\kappa_s}{\kappa_s}\mathbf{n}_s = \boldsymbol{\epsilon}.$$

2.2.21 (i) With the notation in Exercise 2.2.7, the involute of $\boldsymbol{\epsilon}$ is

$$\boldsymbol{\iota}(u) = \boldsymbol{\epsilon} + (\ell - u)\frac{d\boldsymbol{\epsilon}}{du} = \boldsymbol{\gamma} + \frac{1}{\kappa_s}\mathbf{n}_s - (\ell - u)\mathbf{n}_s = \boldsymbol{\gamma} - (\ell - u_0)\mathbf{n}_s,$$

since $u = u_0 - \frac{1}{\kappa_s}$, so $\boldsymbol{\iota}$ is the parallel curve $\boldsymbol{\gamma}^{-(\ell-u_0)}$.
(ii) Using the results of Exercise 2.2.8, the evolute of $\boldsymbol{\iota}$ is

$$\boldsymbol{\iota} + (\ell - s)(-\mathbf{t}) = \boldsymbol{\gamma} + (\ell - s)\mathbf{t} - (\ell - s)\mathbf{t} = \boldsymbol{\gamma}.$$

2.2.22 (i) If $\boldsymbol{\gamma}(\theta) = (x(\theta), y(\theta))$, we have

$$x\cos\theta + y\sin\theta = p.$$

Denoting $d/d\theta$ by a dot, we also have

$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta}.$$

Differentiating the first equation and using the second gives

$$-x\sin\theta + y\cos\theta = \dot{p}.$$

Solving the first and third equations for x and y gives the stated formula for $\boldsymbol{\gamma}$.
(ii) From (i), $\dot{\boldsymbol{\gamma}} = (p + \ddot{p})(-\sin\theta, \cos\theta)$, so $\parallel \dot{\boldsymbol{\gamma}} \parallel = |p + \ddot{p}|$. Hence, $\boldsymbol{\gamma}$ is regular if and only if $p + \ddot{p}$ is never zero. In that case $p + \ddot{p}$ must be either always > 0 or always < 0. But the latter case is impossible since it implies that $\ddot{p} < 0$, contradicting the fact that $\int_0^{2\pi} \ddot{p}\, d\theta = 0$ since p is smooth and 2π-periodic.
(iii) If s is the arc-length of $\boldsymbol{\gamma}$, the signed curvature is $\frac{d}{ds}\left(\theta + \frac{\pi}{2}\right) = \frac{d\theta}{ds} = \frac{1}{ds/d\theta} = \frac{1}{\|\dot{\boldsymbol{\gamma}}\|} = \frac{1}{p+\ddot{p}}$.
(iv) Since $\frac{ds}{d\theta} = p + \ddot{p}$, the length of $\boldsymbol{\gamma}$ is

$$\int_0^{2\pi} (p + \ddot{p})d\theta = \int_0^{2\pi} p\, d\theta.$$

(v) The first part is obvious. The foot of the perpendicular from the origin to the tangent line at $\boldsymbol{\gamma}(\theta)$ is $p(\theta)(\cos\theta, \sin\theta)$, so $w(\theta)$ is the distance between this

point and $p(\theta+\pi)(\cos(\theta+\pi),\sin(\theta+\pi)) = p(\theta+\pi)(-\cos\theta,-\sin\theta)$ which is $p(\theta)+p(\theta+\pi)$.
(vi) The function $q(\theta) = w(\theta) - w(\theta+\pi)$ is smooth and $q(0) = -q(\pi/2)$ because $w(0) = w(\pi)$. Hence, for some angle θ_0 with $0 \leq \theta_0 \leq \pi/2$, we have $q(\theta_0) = 0$, i.e. $w(\theta_0) = w(\theta_0+\pi/2)$. This means that the tangent lines at $\theta_0, \theta_0+\pi/2, \theta_0+\pi$ and $\theta_0+3\pi/2$ form a square.
(vii) From (iv) and (v), the length is

$$\int_0^\pi (p(\theta)+p(\theta+\pi))d\theta = \int_0^\pi w(\theta)\,d\theta,$$

which is equal to πD if $w(\theta)$ is equal to the constant D.
(viii) If $p(\theta)$ is as stated, $w(\theta) = a\left(\cos^2\frac{k\theta}{2} + \cos^2\left(\frac{k\theta}{2}+\frac{k\pi}{2}\right)\right) + 2b = a+2b$.
(ix) If $|k| = 1$ we find that

$$\boldsymbol{\gamma}(\theta) = \left(\frac{1}{2}a, 0\right) + \left(\frac{1}{2}a+b\right)(\cos\theta,\sin\theta),$$

which is the circle with centre $(\frac{1}{2}a, 0)$ and radius $\frac{1}{2}a+b$. In general, $\dot{p} = -\frac{1}{2}ka\sin k\theta$, $\ddot{p} = -\frac{1}{2}k^2a\cos k\theta$, so

$$\kappa_s = \frac{1}{a\cos^2\frac{k\theta}{2} - \frac{1}{2}k^2a\cos k\theta + b} = \frac{1}{\frac{1}{2}a+b+\frac{1}{2}(1-k^2)a\cos k\theta},$$

which is constant $\iff k = 0$ or ± 1 (and $k = 0$ does not give a regular curve).

2.2.23 Parametrize the parabola by $x = \sinh t$, $y = \frac{1}{2}\sinh^2 t$. The arc-length (starting at the origin) is

$$s = \int_0^t (\cosh^2 u + \cosh^2 u \sinh^2 u)^{1/2} du = \frac{1}{2}(t + \sinh t\cosh t).$$

In the notation of Exercise 2.2.10, θ is the angle between $(\cosh t, \cosh t\sinh t)$ and the x-axis, so $\cos\theta = \text{sech}t$, $\sin\theta = \tanh t$, $\mathbf{p} = (0,0)$, $\mathbf{a} = (1,0)$ and we get

$$\boldsymbol{\Gamma} = (s,0) + \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} -\sinh t \\ 1-\frac{1}{2}\sinh^2 t \end{pmatrix}.$$

Using the formulas for s and θ from above gives the stated formula for $\boldsymbol{\Gamma}$ after simplification.

2.2.24 Differentiating $\boldsymbol{\gamma}(t+a) = \boldsymbol{\gamma}(t)$ gives $\mathbf{t}(t+a) = \mathbf{t}(a)$; rotating anti-clockwise by $\pi/2$ gives $\mathbf{n}_s(t+a) = \mathbf{n}_s(t)$; differentiating again gives $\kappa_s(t+a)\mathbf{n}_s(t+a) = \kappa_s(t)\mathbf{n}_s(t)$, so $\kappa_s(t+a) = \kappa_s(t)$.

2.3.1 (i) $\mathbf{t} = (\frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}})$ is a unit vector so $\boldsymbol{\gamma}$ is unit-speed; $\dot{\mathbf{t}} = (\frac{1}{4}(1+t)^{-1/2}, \frac{1}{4}(1-t)^{-1/2}, 0)$, so $\kappa = \| \dot{\mathbf{t}} \| = 1/\sqrt{8(1-t^2)}$; $\mathbf{n} = \frac{1}{\kappa}\dot{\mathbf{t}} = \frac{1}{\sqrt{2}}((1-t)^{1/2}, (1+t)^{1/2}, 0)$; $\mathbf{b} = \mathbf{t} \times \mathbf{n} = (-\frac{1}{2}(1+t)^{1/2}, \frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}})$; $\dot{\mathbf{b}} = (-\frac{1}{4}(1+t)^{-1/2}, -\frac{1}{4}(1-t)^{-1/2}, 0)$ so the torsion $\tau = 1/\sqrt{8(1-t^2)}$. The equation $\dot{\mathbf{n}} = -\kappa\mathbf{t} + \tau\mathbf{n}$ is easily checked.
(ii) $\mathbf{t} = (-\frac{4}{5}\sin t, -\cos t, \frac{3}{5}\sin t)$ is a unit vector so $\boldsymbol{\gamma}$ is unit-speed; $\dot{\mathbf{t}} = (-\frac{4}{5}\cos t, \sin t, \frac{3}{5}\cos t)$, so $\kappa = \| \dot{\mathbf{t}} \| = 1$; $\mathbf{n} = \frac{1}{\kappa}\dot{\mathbf{t}} = (-\frac{4}{5}\cos t, \sin t, \frac{3}{5}\cos t)$; $\mathbf{b} = \mathbf{t} \times \mathbf{n} = (-\frac{3}{5}, 0, -\frac{4}{5})$, so $\dot{\mathbf{b}} = \mathbf{0}$ and $\tau = 0$. By the proof of Proposition 2.3.5, $\boldsymbol{\gamma}$ is a circle of radius $1/\kappa = 1$ with centre $\boldsymbol{\gamma} + \frac{1}{\kappa}\mathbf{n} = (0,1,0)$ in the plane passing through $(0,1,0)$ perpendicular to $\mathbf{b} = (-\frac{3}{5}, 0, -\frac{4}{5})$, i.e. the plane $3x + 4z = 0$.

2.3.2 Let $a = \kappa/(\kappa^2+\tau^2)$, $b = \tau/(\kappa^2+\tau^2)$. By Examples 2.1.3 and 2.3.2, the circular helix with parameters a and b has curvature $\frac{a}{a^2+b^2} = \kappa$ and torsion $\frac{b}{a^2+b^2} = \tau$. By Theorem 2.3.6, every curve with curvature κ and torsion τ is obtained by applying a direct isometry to this helix.

2.3.3 Differentiating $\mathbf{t}.\mathbf{a}$ (= constant) gives $\mathbf{n}.\mathbf{a} = 0$; since $\mathbf{t}, \mathbf{n}, \mathbf{b}$ are an orthonormal basis of $\mathbb{R}^3$, $\mathbf{a} = \mathbf{t}\cos\theta + \mu\mathbf{b}$ for some scalar μ; since $\mathbf{a}$ is a unit vector, $\mu = \pm\sin\theta$; differentiating $\mathbf{a} = \mathbf{t}\cos\theta \pm \mathbf{b}\sin\theta$ gives $\tau = \kappa\cot\theta$. Conversely, if $\tau = \lambda\kappa$, there exists θ with $\lambda = \cot\theta$; differentiating shows that $\mathbf{a} = \mathbf{t}\cos\theta + \mathbf{b}\sin\theta$ is a constant vector and $\mathbf{t}.\mathbf{a} = \cos\theta$ so θ is the angle between $\mathbf{t}$ and $\mathbf{a}$. For the circular helix in Example 2.1.3, the angle between the tangent vector $d\boldsymbol{\gamma}/d\theta = (-a\sin\theta, a\cos\theta, b)$ and the z-axis is the constant $\cos^{-1}(b/\sqrt{a^2+b^2})$.

2.3.4 Differentiating $(\boldsymbol{\gamma}-\mathbf{a}).(\boldsymbol{\gamma}-\mathbf{a}) = r^2$ repeatedly gives $\mathbf{t}.(\boldsymbol{\gamma}-\mathbf{a}) = 0$; $\mathbf{t}.\mathbf{t} + \kappa\mathbf{n}.(\boldsymbol{\gamma}-\mathbf{a}) = 0$, so $\mathbf{n}.(\boldsymbol{\gamma}-\mathbf{a}) = -1/\kappa$; $\mathbf{n}.\mathbf{t} + (-\kappa\mathbf{t}+\tau\mathbf{b}).(\boldsymbol{\gamma}-\mathbf{a}) = \dot{\kappa}/\kappa^2$, and so $\mathbf{b}.(\boldsymbol{\gamma}-\mathbf{a}) = \dot{\kappa}/\tau\kappa^2$; and finally $\mathbf{b}.\mathbf{t} - \tau\mathbf{n}.(\boldsymbol{\gamma}-\mathbf{a}) = (\dot{\kappa}/\tau\kappa^2)^{\cdot}$, and so $\tau/\kappa = (\dot{\kappa}/\tau\kappa^2)^{\cdot}$. Conversely, if Eq. (2.22) holds, then $\rho = -\sigma(\dot{\rho}\sigma)^{\cdot}$, so $(\rho^2 + (\dot{\rho}\sigma)^2)^{\cdot} = 2\rho\dot{\rho} + 2(\dot{\rho}\sigma)(\dot{\rho}\sigma)^{\cdot} = 0$, hence $\rho^2 + (\dot{\rho}\sigma)^2$ is a constant, say r^2 (where $r > 0$). Let $\mathbf{a} = \boldsymbol{\gamma} + \rho\mathbf{n} + \dot{\rho}\sigma\mathbf{b}$; then $\dot{\mathbf{a}} = \mathbf{t} + \dot{\rho}\mathbf{n} + \rho(-\kappa\mathbf{t} + \tau\mathbf{b}) + (\dot{\rho}\sigma)^{\cdot}\mathbf{b} + (\dot{\rho}\sigma)(-\tau\mathbf{n}) = \mathbf{0}$ using Eq. (2.22); so $\mathbf{a}$ is a constant vector and $\| \boldsymbol{\gamma} - \mathbf{a} \|^2 = \rho^2 + (\dot{\rho}\sigma)^2 = r^2$, hence $\boldsymbol{\gamma}$ is contained in the sphere with centre $\mathbf{a}$ and radius r.

2.3.5 $\dot{\boldsymbol{\Gamma}} = P\dot{\boldsymbol{\gamma}}$ so $\mathbf{T} = P\mathbf{t}$ and $\| \dot{\boldsymbol{\Gamma}} \|^2 = (P\dot{\boldsymbol{\gamma}}).(P\dot{\boldsymbol{\gamma}}) = \dot{\boldsymbol{\gamma}}.\dot{\boldsymbol{\gamma}}$ since P is orthogonal. Then, $\ddot{\boldsymbol{\Gamma}} = P\ddot{\boldsymbol{\gamma}}$, taking lengths shows that $\boldsymbol{\gamma}$ and $\boldsymbol{\Gamma}$ have the same curvature κ, and then dividing by κ gives $\mathbf{N} = P\mathbf{n}$. Then $\mathbf{B} = P\mathbf{t} \times P\mathbf{n}$. If P corresponds to a direct isometry (i.e. a rotation), this is equal to $P(\mathbf{t} \times \mathbf{n}) = P\mathbf{b}$, but if P corresponds to an opposite isometry, $P\mathbf{t} \times P\mathbf{n} = -P\mathbf{b}$ (Proposition A.1.6).

2.3.6 Let $\lambda_{ij} = \mathbf{v}_i.\mathbf{v}_j$. The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are orthonormal if and only if $\lambda_{ij} = \delta_{ij}$ (= 1 if $i = j$ and = 0 if $i \neq j$). So it is enough to prove that $\lambda_{ij} = \delta_{ij}$ for all values of s given that it holds for $s = s_0$. Differentiating $\mathbf{v}_i.\mathbf{v}_j$ gives $\dot{\lambda}_{ij} = \sum_{k=1}^{3}(a_{ik}\lambda_{kj} + a_{jk}\lambda_{ik})$. Now $\lambda_{ij} = \delta_{ij}$ is a solution of this system of differential equations because $a_{ij} + a_{ji} = 0$. But the theory of ordinary differential

equations tells us that there is a unique solution with given values when $s = s_0$.

2.3.7 (i) $\mathbf{t} = \dot{\boldsymbol{\gamma}} = (\frac{1}{\sqrt{2}} \sin t\sqrt{2} \cos t\sqrt{2}, \frac{1}{\sqrt{2}})$, $\dot{\mathbf{t}} = (\cos t\sqrt{2}, -\sin t\sqrt{2}, 0)$ so $\kappa = \parallel \dot{\mathbf{t}} \parallel = 1$, $\mathbf{n} = \dot{\mathbf{t}}$, $\mathbf{b} = \mathbf{t} \times \mathbf{n} = (\frac{1}{\sqrt{2}} \sin t\sqrt{2}, \frac{1}{\sqrt{2}} \cos t\sqrt{2}, -\frac{1}{\sqrt{2}})$. Then, $\dot{\mathbf{b}} = \mathbf{n}$ so $\tau = 1$.
(ii) $\mathbf{t} = \dot{\boldsymbol{\gamma}} = (-\frac{1}{\sqrt{3}} \sin t + \frac{1}{\sqrt{2}} \cos t, -\frac{1}{\sqrt{3}} \sin t, -\frac{1}{\sqrt{3}} \sin t - \frac{1}{\sqrt{2}} \cos t)$, so $\kappa = \parallel \dot{\mathbf{t}} \parallel = 1$ and $\mathbf{n} = \dot{\mathbf{t}}$. Then $\mathbf{b} = \mathbf{t} \times \mathbf{n} = (-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$ is constant so $\tau = 0$.

2.3.8 Denoting d/dt by a dot, $\dot{\boldsymbol{\gamma}} = \frac{1}{\sqrt{2}}(\sinh t, \cosh t, 1)$ so $\parallel \dot{\boldsymbol{\gamma}} \parallel = \cosh t$. Hence, the arc-length is $s = \sinh t$, up to adding a constant. The unit tangent vector is $\mathbf{t} = \frac{1}{ds/dt}\dot{\boldsymbol{\gamma}} = \frac{1}{\sqrt{2}}(\tanh t, 1, \operatorname{sech} t)$ so

$$\frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}/dt}{ds/dt} = \frac{\operatorname{sech} t}{\sqrt{2}}(\operatorname{sech}^2 t, 0, -\operatorname{sech} t \tanh t)$$

and $\kappa = \parallel d\mathbf{t}/ds \parallel = \frac{1}{\sqrt{2}}\operatorname{sech}^2 t$. Then, $\mathbf{n} = \frac{1}{\kappa}\frac{d\mathbf{t}}{ds} = (\operatorname{sech} t, 0, -\tanh t)$ and $\mathbf{b} = \mathbf{t} \times \mathbf{n} = \frac{1}{\sqrt{2}}(-\tanh t, 1, -\operatorname{sech} t)$. Finally,

$$\frac{d\mathbf{b}}{ds} = \frac{\operatorname{sech} t}{\sqrt{2}}(-\operatorname{sech}^2 t, 0, \operatorname{sech} t \tanh t) = -\frac{\operatorname{sech}^2 t}{\sqrt{2}}\mathbf{n},$$

so $\tau = \frac{1}{\sqrt{2}}\operatorname{sech}^2 t$.

2.3.9 $\dot{\boldsymbol{\gamma}} = (-\frac{1}{t^2} + 1, 1, -\frac{1}{t^2})$, $\ddot{\boldsymbol{\gamma}} = (\frac{2}{t^3}, 0, \frac{2}{t^3})$, $\dddot{\boldsymbol{\gamma}} = (-\frac{6}{t^4}, 0, -\frac{6}{t^4})$. Hence $\ddot{\boldsymbol{\gamma}}$ is parallel to $\dddot{\boldsymbol{\gamma}}$, and so $\tau = \frac{(\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}}}{\parallel \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} \parallel} = 0$. Alternatively, note that $\frac{1+t^2}{t} - \frac{(1-t)}{t} = t + 1$ so $\boldsymbol{\gamma}$ is contained in the plane $x - z = y$.

2.3.10 We have

$$\begin{aligned} \dot{\boldsymbol{\gamma}} &= (\cos(\ln t) - \sin(\ln t), \sin(\ln t) + \cos(\ln t), 1), \\ \ddot{\boldsymbol{\gamma}} &= \frac{1}{t}(-\sin(\ln t) - \cos(\ln t), \cos(\ln t) - \sin(\ln t), 0), \\ \therefore \quad \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} &= \frac{1}{t}(\sin(\ln t) - \cos(\ln t), -\sin(\ln t) - \cos(\ln t), 2), \end{aligned}$$

so $\kappa = \frac{\parallel \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} \parallel}{\parallel \dot{\boldsymbol{\gamma}} \parallel^3} = \frac{1}{t}\sqrt{\frac{2}{9}}$.

2.3.11 The torsion is defined whenever the curvature is non-zero. In that case $\parallel \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} \parallel > 0$ for all t and so is a smooth function of t (compare the proof of Proposition 1.3.5). Hence, $\tau = \frac{(\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}}}{\parallel \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} \parallel}$ is smooth.

2.3.12 Let a dot denote d/dt. Then, $\dot{\boldsymbol{\delta}} = \ddot{\boldsymbol{\gamma}} = \kappa\mathbf{n}$, so the unit tangent vector of $\boldsymbol{\delta}$ is $\mathbf{T} = \mathbf{n}$ and its arc-length s satisfies $ds/dt = \kappa$. Now $d\mathbf{T}/ds = \dot{\mathbf{n}}/(ds/dt) = \kappa^{-1}(-\kappa\mathbf{t} + \tau\mathbf{b}) = -\mathbf{t} + \frac{\tau}{\kappa}\mathbf{b}$. Hence, the curvature of $\boldsymbol{\delta}$ is $\parallel -\mathbf{t} + \frac{\tau}{\kappa}\mathbf{b} \parallel = (1 + \frac{\tau^2}{\kappa^2})^{1/2} =$

μ, say. The principal normal of $\boldsymbol{\delta}$ is $\mathbf{N} = \mu^{-1}(-\mathbf{t} + \frac{\tau}{\kappa}\mathbf{b})$ and its binormal is $\mathbf{B} = \mathbf{T} \times \mathbf{N} == \mu^{-1}(\mathbf{b} + \frac{\tau}{\kappa}\mathbf{t})$. The torsion T of $\boldsymbol{\delta}$ is given by $d\mathbf{B}/ds = -T\mathbf{N}$, i.e. $\kappa^{-1}\dot{\mathbf{B}} = -T\mathbf{N}$. Computing the derivatives and equating coefficients of $\mathbf{b}$ gives $T = (\kappa\dot{\tau} - \tau\dot{\kappa})/\kappa(\kappa^2 + \tau^2)$.

2.3.13 The curve is $\boldsymbol{\gamma}(t) = (e^{\lambda t}\cos t, e^{\lambda t}\sin t, e^{\lambda t})$. The angle between the tangent vector $\dot{\boldsymbol{\gamma}} = e^{\lambda t}(\lambda\cos t - \sin t, \lambda\sin t + \cos t, \lambda)$ and the vector $\mathbf{k} = (0,0,1)$ is $\cos^{-1}\left(\frac{\dot{\boldsymbol{\gamma}}.\mathbf{k}}{\|\dot{\boldsymbol{\gamma}}\|}\right) = \cos^{-1}\frac{\lambda}{\sqrt{2\lambda^2+1}}$, which is constant.

2.3.14 We find that

$$\kappa = \frac{2(9b^2c^2t^4 + 9a^2c^2t^2 + a^2b^2)^{1/2}}{(a^2 + 4b^2t^2 + 9c^2t^4)^{3/2}}, \quad \tau = \frac{3abc}{9b^2c^2t^4 + 9a^2c^2t^2 + a^2b^2}.$$

Now $\boldsymbol{\gamma}$ is a generalized helix if and only if there is a constant λ such that $\tau = \lambda\kappa$, i.e.

$$3abc(a^2 + 4b^2t^2 + 9c^2t^4)^{3/2} = 2\lambda(9b^2c^2t^4 + 9a^2c^2t^2 + a^2b^2)^{3/2},$$

which is equivalent to

$$(*) \qquad 9b^2c^2t^4 + 9a^2c^2t^2 + a^2b^2 = \mu(a^2 + 4b^2t^2 + 9c^2t^4)$$

where $\mu = (3abc/2\lambda)^{2/3}$. The last equation holds if and only if

$$9b^2c^2 = 9\mu c^2,\ 9a^2c^2 = 4\mu b^2,\ a^2b^2 = \mu a^2.$$

The first and third equations give $\mu = b^2$ and the second gives $\mu = 9a^2c^2/4b^2$. Hence, if $\boldsymbol{\gamma}$ is a generalized helix $9a^2c^2/4b^2 = b^2$, which is equivalent to the stated condition. Conversely, if this condition holds, taking $\mu = b^2$ (and hence $\lambda = 3ac/2b^2$) satisfies (*) so $\tau = \frac{3ac}{2b^2}\kappa$ and $\boldsymbol{\gamma}$ is a generalized helix.

2.3.15 (i) From the definition, $\tilde{\boldsymbol{\gamma}}(\tilde{s}) = \boldsymbol{\gamma}(s) + a(s)\mathbf{n}(s)$, where a is a scalar possibly depending on s. Then,

$$\frac{d\tilde{s}}{ds}\tilde{\mathbf{t}} = \frac{d\boldsymbol{\gamma}}{ds} + \frac{da}{ds}\mathbf{n} + a\frac{d\mathbf{n}}{ds} = (1 - \kappa a)\mathbf{t} + \frac{da}{ds}\mathbf{n} + \tau a\mathbf{b}.$$

But we are given that $\tilde{\mathbf{n}} = \pm\mathbf{n}$, so $\tilde{\mathbf{t}}$ is perpendicular to $\mathbf{n}$. This implies that $da/ds = 0$ and hence a is constant.

(ii) We have

$$\frac{d}{ds}(\mathbf{t}.\tilde{\mathbf{t}}) = \kappa\mathbf{n}.\tilde{\mathbf{t}} + \frac{d\tilde{s}}{ds}\tilde{\kappa}\mathbf{t}.\tilde{\mathbf{n}} = 0,$$

since $\tilde{\mathbf{t}}$ is perpendicular to $\tilde{\mathbf{n}}$ and hence to $\mathbf{n}$, and similarly $\mathbf{t}$ is perpendicular to $\tilde{\mathbf{n}}$. Hence, $\mathbf{t}.\tilde{\mathbf{t}}$ is a constant, say $\cos\alpha$ (it must lie between -1 and 1 as $\mathbf{t}$ and $\tilde{\mathbf{t}}$ are unit

vectors). Then, since $\mathbf{t}$ is perpendicular to $\mathbf{n}$, we must have $\tilde{\mathbf{t}} = \cos\alpha\mathbf{t} \pm \sin\alpha\mathbf{b}$, and we can assume the sign is $-$ by changing α to $-\alpha$ if necessary. Then,

$$\tilde{\mathbf{b}} = \tilde{\mathbf{t}} \times \tilde{\mathbf{n}} = \pm(\cos\alpha\mathbf{t} - \sin\alpha\mathbf{b}) \times \mathbf{n} = \pm(\sin\alpha\mathbf{t} + \cos\alpha\mathbf{b}).$$

(iii) From (i) and (ii),

$$\frac{d\tilde{s}}{ds}(\cos\alpha\mathbf{t} - \sin\alpha\mathbf{b}) = (1 - \kappa a)\mathbf{t} + \tau a\mathbf{b}.$$

Equating coefficients of $\mathbf{t}$ and $\mathbf{b}$ gives the first pair of equations. The second pair follows by interchanging the roles of $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$, which changes α to $-\alpha$ and a to $-a$.
(iv) This follows from the first pair of equations in (iii).
(v) The first equation follows from the two equations in (iii) involving the torsion, noting that $\frac{d\tilde{s}}{ds}\frac{ds}{d\tilde{s}} = 1$. The first equation follows similarly from the other two equations in (iii).

2.3.16 Let $\boldsymbol{\gamma}(s)$ be a unit-speed plane curve, and let $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} + a\mathbf{n}$ for any constant $a \neq 0$. If $\tilde{s}$ is the arc-length of $\tilde{\boldsymbol{\gamma}}$, the unit tangent vector $\tilde{\mathbf{t}}$ of $\tilde{\boldsymbol{\gamma}}$ is given by $\frac{d\tilde{s}}{ds}\tilde{\mathbf{t}} = (1 - \kappa a)\mathbf{t}$. Hence $\tilde{\mathbf{t}} = \pm\mathbf{t}$ and so (since $\boldsymbol{\gamma}$ is planar) $\tilde{\mathbf{n}} = \pm\mathbf{n}$.

2.3.17 The 'only if' part was proved in Exercise 2.3.15(iv). Conversely, if $a\kappa + b\tau = 1$, let $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} + a\mathbf{n}$. Then,

$$\frac{d\tilde{s}}{ds}\tilde{\mathbf{t}} = (1 - \kappa a)\mathbf{t} + \tau a\mathbf{b} = \tau(b\mathbf{t} + a\mathbf{b}).$$

Hence, $\tilde{\mathbf{t}} = \pm\frac{1}{\sqrt{a^2+b^2}}(b\mathbf{t} + a\mathbf{b})$. Differentiating,

$$\tilde{\kappa}\tilde{\mathbf{n}} = \pm\frac{1}{\sqrt{a^2+b^2}}\frac{ds}{d\tilde{s}}(b\frac{d\mathbf{t}}{ds} + a\frac{d\mathbf{b}}{ds}) = \pm\frac{\kappa b - \tau a}{\sqrt{a^2+b^2}}\frac{ds}{d\tilde{s}}\mathbf{n}.$$

It follows that $\tilde{\mathbf{n}} = \pm\mathbf{n}$.

2.3.18 The solutions of Exercises 2.3.15 and 2.3.17 show that $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} + a\mathbf{n}$ is a Bertrand mate of $\boldsymbol{\gamma}$ (a being a constant) if and only if $a\kappa + b\tau = 1$ for some constant b, and that such curves $\tilde{\boldsymbol{\gamma}}$ are all the Bertrand mates of $\boldsymbol{\gamma}$. Thus, $\boldsymbol{\gamma}$ has more than one Bertrand mate if and only if the equation $a\kappa + b\tau = 1$ is satisfied by more than one value of a (and some corresponding values of b). In that case κ and τ must be constant, so $\boldsymbol{\gamma}$ is a circular helix (Exercise 2.3.2). To see that $\tilde{\boldsymbol{\gamma}}$ is then a circular helix with the same axis and pitch as $\boldsymbol{\gamma}$, take $\boldsymbol{\gamma}$ to have axis the z-axis (this can be achieved by applying an isometry of $\mathbb{R}^3$). Then, $\boldsymbol{\gamma}(t) = (c\cos\lambda t, c\sin\lambda t, d\lambda t)$, where c is the radius of $\boldsymbol{\gamma}$, $2\pi d$ is its pitch and $\lambda = (c^2 + d^2)^{-1/2}$. Hence, $\mathbf{n} = (-\cos\lambda t, -\sin\lambda t, 0)$ so the most general Bertrand mate of $\boldsymbol{\gamma}$ is

$$\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma}_a + \mathbf{n} = ((c-a)\cos\lambda t, (c-a)\sin\lambda t, d\lambda t)\,,$$

which is a circular helix with axis the z-axis, pitch $2\pi d$ and radius $|c-a|$.

2.3.19 If κ is constant, Eq. (2.22) in Exercise 2.3.4 gives $\tau = 0$, so the curve is a circle (constant curvature and zero torsion).

2.3.20 Let $\boldsymbol{\gamma}(t)$ be a unit-speed parametrization of $\mathcal{C}$. If all the normal planes of $\mathcal{C}$ pass through a point $\mathbf{p}$, there are scalars λ, μ (possibly depending on t) such that

$$\boldsymbol{\gamma} + \lambda\mathbf{n} + \mu\mathbf{b} = \mathbf{p}$$

for all t. Differentiating and using the Frenet-Serret equations,

$$\mathbf{t} + \dot{\lambda}\mathbf{n} + \lambda(-\kappa\mathbf{t} + \tau\mathbf{b}) + \dot{\mu}\mathbf{b} - \mu\tau\mathbf{n} = \mathbf{0}.$$

Hence,

$$1 - \lambda\kappa = 0, \quad \dot{\lambda} - \mu\tau = 0, \quad \lambda\tau + \dot{\mu} = 0.$$

Using the first and third equations,

$$\frac{d}{ds}(\lambda^2 + \mu^2) = 2\lambda\dot{\lambda} + 2\mu\dot{\mu} = 2\lambda(\mu\tau) + 2\mu(-\lambda\tau) = 0,$$

so $\parallel \boldsymbol{\gamma} - \mathbf{p} \parallel = \sqrt{\lambda^2 + \mu^2}$ is constant. Hence, $\boldsymbol{\gamma}$ lies on a sphere with centre $\mathbf{p}$. (An alternative is to make use of Exercise 2.3.4.)

2.3.21 (i) is obvious.
(ii) $\dot{F} = \dot{\boldsymbol{\gamma}}.\mathbf{N}$ so $\dot{F}(t_0) = 0$ if and only if the tangent vector of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t_0)$ is parallel to Π. Using (i) this gives (ii).
(iii) $\ddot{F} = \ddot{\boldsymbol{\gamma}}.\mathbf{N} = \kappa\mathbf{n}.\mathbf{N}$, so $\ddot{F}(t_0) = 0$ if and only if $\mathbf{n}(t_0)$ is parallel to Π. Thus, the unique plane Π such that $F(t_0) = \dot{F}(t_0) = \ddot{F}(t_0) = 0$ is that passing through $\boldsymbol{\gamma}(t_0)$ and parallel to $\mathbf{t}(t_0)$ and $\mathbf{n}(t_0)$, i.e. perpendicular to $\mathbf{b}(t_0)$.
(iv) If $\boldsymbol{\gamma}$ is contained in the plane $\mathbf{v}.\mathbf{N}' = d'$, then $\dot{\boldsymbol{\gamma}}.\mathbf{N}' = \ddot{\boldsymbol{\gamma}}.\mathbf{N}' = 0$ so by (iii) the osculating plane Π of $\boldsymbol{\gamma}$ at any point $\boldsymbol{\gamma}(t_0)$ is perpendicular to $\mathbf{N}'$, and so is parallel to Π'. But both Π and Π' pass through $\boldsymbol{\gamma}(t_0)$, so they must coincide.
(v) The (signed) distance from $\boldsymbol{\gamma}(t)$ to Π is $d(t) = (\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(t_0)).\mathbf{b}(t_0)$. Then, $d(t_0) = 0$ and

$$\begin{aligned}
\dot{d}(t_0) &= \mathbf{t}(t_0).\mathbf{b}(t_0) = 0, \\
\ddot{d}(t_0) &= \kappa(t_0)\mathbf{n}(t_0).\mathbf{b}(t_0) = 0, \\
\dddot{d}(t_0) &= (\dot{\kappa}\mathbf{n} + \kappa(-\kappa\mathbf{t} + \tau\mathbf{b}))(t_0).\mathbf{b}(t_0) = \kappa(t_0)\tau(t_0).
\end{aligned}$$

If $\tau(t_0) \neq 0$, Taylor's theorem gives

$$d(t) = \frac{1}{6}\kappa(t_0)\tau(t_0)(t-t_0)^3 + \text{higher order terms}.$$

As t passes from values $< t_0$ to values $> t_0$, the leading term changes sign, so $\boldsymbol{\gamma}$ crosses Π.

2.3.22 For the circular helix $\boldsymbol{\gamma}(t) = (a\cos t, a\sin t, bt)$, $\mathbf{b} = \frac{1}{\sqrt{a^2+b^2}}(b\sin t, -b\cos t, a)$, so by Exercise 2.3.21 the osculating plane at $\boldsymbol{\gamma}(t_0)$ is $(\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(t_0)).\mathbf{b}(t_0) = 0$, i.e.

$$xb\sin t_0 - yb\cos t_0 + az = abt_0.$$

2.3.23 Proceeding as in the preceding exercise, the osculating plane at $\boldsymbol{\gamma}(t)$ is found to be

$$3t^2x - 3ty + z = t^3.$$

If P_1, P_2, P_3 correspond to $t = t_1, t_2, t_3$, the three osculating planes intersect at a single point if and only if the determinant

$$\begin{vmatrix} 3t_1^2 & -3t_1 & 1 \\ 3t_2^2 & -3t_2 & 1 \\ 3t_3^2 & -3t_3 & 1 \end{vmatrix} \neq 0.$$

But the determinant equals $9(t_1-t_2)(t_2-t_3)(t_3-t_1)$, and this is indeed non-zero as t_1, t_2, t_3 are distinct.

If Q is the point (X, Y, Z), then t_1, t_2, t_3 are the roots of

$$t^3 - 3t^2X + 3tY - Z = 0.$$

On the other hand, if the plane Π passing through the points $\boldsymbol{\gamma}(t_1), \boldsymbol{\gamma}(t_2)$ and $\boldsymbol{\gamma}(t_3)$ has equation

$$ax + by + cz + d = 0,$$

then t_1, t_2, t_3 are also the roots of

$$at + bt^2 + ct^3 + d = 0.$$

The two cubic equations must be the same, so Π must be the plane

$$3Yx - 3Xy + z = Z,$$

and the point (X, Y, Z) obviously lies on this plane.

2.3.24 We can assume that $\boldsymbol{\gamma}$ is unit-speed. If all the osculating planes pass through a point $\mathbf{p}$, there are scalars λ, μ such that

$$\boldsymbol{\gamma} + \lambda\mathbf{t} + \mu\mathbf{n} = \mathbf{p}.$$

Proceeding as in Exercise 2.3.20 we find that

$$\mu\tau = 0, \quad \dot{\lambda} - \mu\kappa = -1, \quad \lambda\kappa + \dot{\mu} = 0.$$

If $\mu = 0$ the third equation gives $\lambda = 0$, which contradicts the second equation. So $\mu \neq 0$ and the first equation gives $\tau = 0$, i.e. $\boldsymbol{\gamma}$ is planar.

2.3.25 Let $\boldsymbol{\gamma}$ be a unit-speed curve. Its orthogonal projection onto the normal plane at $\boldsymbol{\gamma}(t_0)$ is

$$\boldsymbol{\Gamma} = \boldsymbol{\gamma} - (\boldsymbol{\gamma}.\mathbf{t}(t_0))\mathbf{t}(t_0).$$

Then,

$$\dot{\boldsymbol{\Gamma}} = \mathbf{t} - (\mathbf{t}.\mathbf{t}(t_0))\mathbf{t}(t_0) = 0 \quad \text{if } t = t_0.$$

Hence, $\boldsymbol{\Gamma}$ has a singular point at $t = t_0$. Similarly, we find that

$$\ddot{\boldsymbol{\Gamma}}(t_0) = \kappa(t_0)\mathbf{n}(t_0), \ \dddot{\boldsymbol{\Gamma}}(t_0) = \dot{\kappa}(t_0)\mathbf{n}(t_0) + \kappa(t_0)\tau(t_0)\mathbf{t}(t_0).$$

These vectors are linearly independent if $\kappa(t_0)$ and $\tau(t_0)$ are non-zero, so in that case $\boldsymbol{\Gamma}$ has an ordinary cusp at $t = t_0$.

The orthogonal projection of $\boldsymbol{\gamma}$ onto its osculating plane at $\boldsymbol{\gamma}(t_0)$ is

$$\tilde{\boldsymbol{\Gamma}} = \boldsymbol{\gamma} - (\boldsymbol{\gamma}.\mathbf{b}(t_0))\mathbf{b}(t_0),$$

so

$$\dot{\tilde{\boldsymbol{\Gamma}}}(t_0) = \mathbf{t}(t_0) - (\mathbf{t}(t_0).\mathbf{b}(t_0))\mathbf{b}(t_0) = \mathbf{t}(t_0) \neq \mathbf{0},$$

so $\boldsymbol{\gamma}(t_0)$ is a regular point of $\tilde{\boldsymbol{\Gamma}}$.

2.3.26 (i) is obvious.

(ii) $\dot{F} = 4\mathbf{t}.(\boldsymbol{\gamma} - \mathbf{c})$ so $\dot{F}(t_0) = 0$ if and only if $\mathbf{t}(t_0)$ is perpendicular to $\boldsymbol{\gamma}(t_0) - \mathbf{c}$, i.e. $\boldsymbol{\gamma}$ is tangent to $\mathcal{S}$ at $\boldsymbol{\gamma}(t_0)$. Next,

$$\ddot{F} = 2\kappa\mathbf{n}.(\boldsymbol{\gamma} - \mathbf{c}) + 2, \ \dddot{F} = 2\dot{\kappa}\mathbf{n}.(\boldsymbol{\gamma} - \mathbf{c}) + 2\kappa(-\kappa\mathbf{t} + \tau\mathbf{b}).(\boldsymbol{\gamma} - \mathbf{c}).$$

Hence, $F(t_0) = \dot{F}(t_0) = \ddot{F}(t_0) = \dddot{F}(t_0) = 0$ if and only if $\boldsymbol{\gamma}$ is tangent to $\mathcal{S}$ at $\boldsymbol{\gamma}(t_0)$ and

$$\kappa\mathbf{n}.(\boldsymbol{\gamma} - \mathbf{c}) = -1, \quad \kappa\tau\mathbf{b}.(\boldsymbol{\gamma} - \mathbf{c}) = \frac{\dot{\kappa}}{\kappa}$$

(all quantities evaluated at $t = t_0$). These equations and the fact that $\boldsymbol{\gamma}(t_0) - \mathbf{c}$ is perpendicular to $\mathbf{t}(t_0)$ are equivalent to

$$\boldsymbol{\gamma} - \mathbf{c} = -\frac{1}{\kappa}\mathbf{n} + \frac{\dot{\kappa}}{\kappa^2\tau}\mathbf{b},$$

so the centre $\mathbf{c}$ of $\mathcal{S}$ is uniquely determined and its radius is

$$R = \parallel \boldsymbol{\gamma} - \mathbf{c} \parallel = \sqrt{\frac{1}{\kappa^2} + \frac{\dot{\kappa}^2}{\kappa^4\tau^2}}$$

(again, all quantities evaluated at $t = t_0$). The centre $\mathbf{c}$ is independent of t_0 if and only if

$$\frac{d}{dt}\left(\boldsymbol{\gamma} + \frac{1}{\kappa}\mathbf{n} - \frac{\dot{\kappa}}{\kappa^2\tau}\mathbf{b}\right) = 0$$

when $t = t_0$. Using Frenet-Serret, the derivative is found to be

$$\left(\frac{\tau}{\kappa} - \frac{d}{dt}\left(\frac{\dot{\kappa}}{\kappa^2\tau}\right)\right)\mathbf{b}.$$

By Exercise 2.3.4, $\mathbf{c}$ is independent of t_0 if and only if $\boldsymbol{\gamma}$ is spherical, in which case the solution of Exercise 2.3.4 shows that $\boldsymbol{\gamma}$ lies on the sphere with centre $\mathbf{c}$ and radius R, i.e. the osculating sphere.

2.3.27 Since $\mathbf{b}(t_0)$ is perpendicular to the osculating plane at $\boldsymbol{\gamma}(t_0)$, the centre of the osculating circle will be of the form $\mathbf{c}' = \mathbf{c}(t_0) + \lambda\mathbf{b}(t_0)$ for some scalar λ. As this point lies on the osculating plane, it must be of the form $\boldsymbol{\gamma}(t_0) + \mu\mathbf{t}(t_0) + \nu\mathbf{n}(t_0)$ for some μ, ν. Hence,

$$\boldsymbol{\gamma} + \frac{1}{\kappa}\mathbf{n} - \frac{\dot{\kappa}}{\kappa^2\tau}\mathbf{b} + \lambda\mathbf{b} = \boldsymbol{\gamma} + \mu\mathbf{t} + \nu\mathbf{n},$$

all quantities being evaluated at $t = t_0$. So $\lambda = \frac{\dot{\kappa}}{\kappa^2\tau}$ and the centre of the osculating circle is

$$\mathbf{c}(t_0) + \lambda\mathbf{b}(t_0) = \boldsymbol{\gamma}(t_0) + \frac{1}{\kappa(t_0)}\mathbf{n}(t_0).$$

If R is the radius of the osculating sphere, the radius r of the osculating circle is given by

$$r^2 = R^2 - \parallel \mathbf{c}' - \mathbf{c} \parallel^2 = \frac{1}{\kappa^2} + \frac{\dot{\kappa}^2}{\kappa^4\tau^2} - \lambda^2 = \frac{1}{\kappa^2},$$

so $r = 1/\kappa(t_0)$.

Chapter 3

3.1.1 $\dot{\boldsymbol{\gamma}} = (-\sin t - a\sin 2t, \cos t + a\cos 2t)$, $\parallel \dot{\boldsymbol{\gamma}} \parallel^2 = 1 + a^2 + 2a\cos t \geq 1 + a^2 - 2|a| = (1 - |a|)^2$ so $\boldsymbol{\gamma}$ is regular if $|a| \neq 1$. If $|a| = 1$ then $\parallel \dot{\boldsymbol{\gamma}} \parallel = 2(1 + a\cos t)$ so the origin is a singular point of $\boldsymbol{\gamma}$. If $a = 0$ then $\boldsymbol{\gamma}$ is a circle. If $0 < |a| < 1$, then $\boldsymbol{\gamma}(t_1) = \boldsymbol{\gamma}(t_2) \Longrightarrow 1 + a\cos t_1 = 1 + a\cos t_2 \Longrightarrow \cos t_1 = \cos t_2 \Longrightarrow t_2 = t_1$ or $2\pi - t_1$. In the latter case, $\boldsymbol{\gamma}(t_2) = ((1 + a\cos t_1)\cos t_1, -(1 + a\cos t_1)\sin t_1)$ so $\boldsymbol{\gamma}(t_1) = \boldsymbol{\gamma}(t_2) \Longrightarrow \sin t_1 = 0 \Longrightarrow t_1 = 0$ or π. In all cases, $t_2 - t_1$ is a multiple of 2π, so $\boldsymbol{\gamma}$ is a closed curve with period 2π without self-intersections. If $|a| > 1$, $\boldsymbol{\gamma}$ passes through the origin when $\cos t = -1/a$, which has two roots

with $0 \le t < 2\pi$, say $t_1 < t_2$, so the origin is a self-intersection. The picture is qualitatively similar to that in Example 1.1.7 (which is the case $a = 2$), so the complement of the image of $\boldsymbol{\gamma}$ is the union of two bounded regions enclosed by the part of the curve with $t_1 \le t \le t_2$, and an unbounded region.

3.1.2 $\dot{\boldsymbol{\gamma}} = (-\sin t - 2a \sin t \cos t, \cos t + a(\cos^2 t - \sin^2 t))$ so

$$\tan\varphi = -\frac{\cos t + a\cos 2t}{\sin t + a \sin 2t},$$

$$\therefore \quad \sec^2 t \frac{d\varphi}{dt} = \frac{1 + 3a\cos t + 2a^2}{(\sin t + a\sin 2t)^2},$$

$$\therefore \quad \frac{d\varphi}{dt} = \frac{1 + 3a\cos t + 2a^2}{1 + 2a\cos t + a^2} = 1 + \frac{a(\cos t + a)}{1 + 2a\cos t + a^2}.$$

If $|a| < 1$, $\boldsymbol{\gamma}$ is a simple closed curve of period 2π (Exercise 3.1.1), so by the Umlaufsatz,

$$\int_0^{2\pi} \frac{d\varphi}{dt} dt = 2\pi,$$

giving

$$\int_0^{2\pi} \frac{a(\cos t + a)}{1 + 2a\cos t + a^2} dt = 0.$$

If $|a| > 1$, it is clear geometrically that the tangent vector of $\boldsymbol{\gamma}$ rotates by 4π on going once around $\boldsymbol{\gamma}$ (see the diagram in Example 1.1.7 for the case $a = 2$), so

$$\int_0^{2\pi} \frac{d\varphi}{dt} dt = 4\pi,$$

giving

$$\int_0^{2\pi} \frac{a(\cos t + a)}{1 + 2a\cos t + a^2} dt = 2\pi.$$

3.2.1 By Appendix 1, any isometry M of $\mathbb{R}^2$ is of the form $M(\mathbf{v}) = P\mathbf{v} + \mathbf{b}$, where P is a 3×3 orthogonal matrix and $\mathbf{b}$ is a constant vector. If $\tilde{\boldsymbol{\gamma}} = M(\boldsymbol{\gamma})$, then $\dot{\tilde{\boldsymbol{\gamma}}} = P\dot{\boldsymbol{\gamma}}$, so $\parallel \dot{\tilde{\boldsymbol{\gamma}}} \parallel = \parallel \dot{\boldsymbol{\gamma}} \parallel$, which implies that $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ have the same length. If we think of $\boldsymbol{\gamma}$ as a curve in the xy-plane in $\mathbb{R}^3$, Eq. (3.2) can be written $\mathcal{A}(\boldsymbol{\gamma}) = \int_0^T (\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}).\mathbf{k}\, dt$, where $\mathbf{k} = (0, 0, 1)$. It now follows from Proposition A.1.6 that $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ have the same area (note that if M is opposite, the area appears to change sign, but it does not because in that case $\tilde{\boldsymbol{\gamma}}$ is negatively-oriented when $\boldsymbol{\gamma}$ is positively-oriented).

3.2.2 Parametrizing the ellipse by $\boldsymbol{\gamma}(t) = (p\cos t, q\sin t)$, with $0 \le t \le 2\pi$, its area is $\int_0^{2\pi} \sqrt{pq\sin^2 t + pq\cos^2 t}\, dt = 2\pi\sqrt{pq}$. By the isoperimetric inequality, the length ℓ of the ellipse satisfies $\ell \ge \sqrt{4\pi \times \pi pq} = 2\pi\sqrt{pq}$, with equality if and only

if the ellipse is a circle, i.e. $p = q$. But its length is $\int_0^{2\pi} \parallel \dot{\boldsymbol{\gamma}} \parallel dt = \int_0^{2\pi} \sqrt{p^2 \sin^2 t + q^2 \cos^2 t}\, dt$.

3.2.3 $\boldsymbol{\gamma}(t') = \boldsymbol{\gamma}(t) \iff \cos t' = \cos t$ and $\sin t' = \sin t \iff t' - t$ is a multiple of 2π, so $\boldsymbol{\gamma}$ is simple closed with period 2π. Taking $x = p\cos t, y = q \sin t$ in Eq. 3.2 gives the area as $\frac{1}{2}\int_0^{2\pi} pq\, dt = \pi pq$.

3.3.1 Let (x_1, y_1) and (x_2, y_2) be points in the interior of the ellipse, so that $\frac{x_i^2}{p^2} + \frac{y_i^2}{q^2} < 1$ for $i = 1, 2$. A point of the line segment joining the two points is

$$(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2)$$

for some $0 \leq t \leq 1$. This is in the interior of the ellipse because

$$\begin{aligned}
&\frac{(tx_1 + (1-t)x_2)^2}{p^2} + \frac{(ty_1 + (1-t)y_2)^2}{q^2} \\
&= t^2\left(\frac{x_1^2}{p^2} + \frac{y_1^2}{q^2}\right) + (1-t)^2\left(\frac{x_2^2}{p^2} + \frac{y_2^2}{q^2}\right) + 2t(1-t)\left(\frac{x_1x_2}{p^2} + \frac{y_1y_2}{q^2}\right) \\
&< t^2 + (1-t)^2 + t(1-t)\left(\frac{x_1^2}{p^2} + \frac{y_1^2}{q^2} + \frac{x_2^2}{p^2} + \frac{y_2^2}{q^2}\right) \leq t^2 + (1-t)^2 + 2t(1-t) = 1.
\end{aligned}$$

3.3.2 $\dot{\boldsymbol{\gamma}} = (-\sin t - 2\sin 2t, \cos t + 2\cos 2t)$ and $\parallel \dot{\boldsymbol{\gamma}} \parallel = \sqrt{5 + 4\cos t}$, so the angle φ between $\dot{\boldsymbol{\gamma}}$ and the x-axis is given by $\cos\varphi = \frac{-\sin t - 2\sin 2t}{\sqrt{5+4\cos t}}$, $\sin\varphi = \frac{\cos t + 2\cos 2t}{\sqrt{5+4\cos t}}$. Differentiating the second equation gives $\dot{\varphi}\cos\varphi = \frac{-\sin t(24\cos^2 t + 42\cos t + 9)}{(5+4\cos t)^{3/2}}$, so $\dot{\varphi} = \frac{\sin t(24\cos^2 t + 42\cos t + 9)}{(5+4\cos t)(\sin t + 2\sin 2t)} = \frac{9+6\cos t}{5+4\cos t}$. Hence, if s is the arc-length of $\boldsymbol{\gamma}$, $\kappa_s = d\varphi/ds = (d\varphi/dt)/(ds/dt) = (9 + 6\cos t)/(5 + 4\cos t)^{3/2}$, so

$$\dot{\kappa}_s = \frac{12\sin t(2 + \cos t)}{(5 + 4\cos t)^{5/2}}.$$

This vanishes at only two points of the curve, where $t = 0$ and $t = \pi$.

3.3.3 From $\boldsymbol{\epsilon}(s) = \boldsymbol{\gamma}(s) + \frac{1}{\kappa_s}\mathbf{n}_s$ we get $\dot{\boldsymbol{\epsilon}} = -\dot{\kappa}_s\mathbf{n}_s/\kappa_s^2$, so $\boldsymbol{\epsilon}$ has a singular point where $\dot{\kappa}_s = 0$, i.e. where $\boldsymbol{\gamma}$ has a vertex.

3.3.4 Parametrize by $\boldsymbol{\gamma}(x) = (x, f(x))$ and denote d/dx by a dot. The arc-length s is given by $\frac{ds}{dx} = \sqrt{1 + \dot{f}^2}$, so the unit tangent vector is $\mathbf{t} = \frac{1}{\sqrt{1+\dot{f}^2}}(1, \dot{f})$ and we find that

$$\frac{d\mathbf{t}}{ds} = \frac{1}{ds/dx}\dot{\mathbf{t}} = \frac{1}{(1 + \dot{f}^2)^2}(-\dot{f}\ddot{f}, \ddot{f}).$$

The signed unit normal $\mathbf{n}_s = \frac{1}{\sqrt{1+\dot{f}^2}}(-\dot{f}, 1)$, so the signed curvature is $\kappa_s = \frac{\ddot{f}}{(1+\dot{f}^2)^{3/2}}$. Then,

$$\dot{\kappa}_s = \frac{\dddot{f}(1+\dot{f}^2) - 3\dot{f}\ddot{f}^2}{(1+\dot{f}^2)^{5/2}}.$$

Hence, $\frac{d\kappa_s}{ds} = 0 \Longleftrightarrow \frac{d\kappa_s}{dx} = 0 \Longleftrightarrow \dddot{f}(1+\dot{f}^2) = 3\dot{f}\ddot{f}^2$.

3.3.5 $\dot{\boldsymbol{\gamma}} = (a - b\cos t, b\sin t)$, so if φ is a turning angle for $\boldsymbol{\gamma}$,

$$\tan\varphi = \frac{b\sin t}{a - b\cos t},$$

$$\therefore \quad \dot{\varphi}\sec^2\varphi = \frac{b(a\cos t - b)}{(a - b\cos t)^2},$$

$$\therefore \quad \dot{\varphi} = \frac{b(a\cos t - b)}{(a - b\cos t)^2}\frac{(a - b\cos t)^2}{(a - b\cos t)^2 + b^2\sin^2 t} = \frac{b(a\cos t - b)}{a^2 - 2ab\cos t + b^2}.$$

If s is the arc-length of $\boldsymbol{\gamma}$, the signed curvature is

$$\kappa_s = \frac{d\varphi}{ds} = \frac{d\varphi/dt}{ds/dt} = \frac{\dot{\varphi}}{\parallel \dot{\boldsymbol{\gamma}} \parallel} = \frac{b(a\cos t - b)}{(a^2 - 2ab\cos t + b^2)^{3/2}}.$$

Hence,

$$\frac{d\kappa_s}{dt} = \frac{ab\sin t(2b^2 - a^2 - ab\sin t\cos t)}{(a^2 - 2ab\cos t + b^2)^{5/2}}.$$

Thus, $\boldsymbol{\gamma}(t)$ is a vertex if and only if $\sin t = 0$ or $\cos t = \frac{2b^2-a^2}{ab}$. Hence, there are vertices when t is an integer multiple of π, and no other vertices if $\left|\frac{2b^2-a^2}{ab}\right| > 1$. On the other hand, if $\left|\frac{2b^2-a^2}{ab}\right| \leq 1$, i.e. if $\frac{a-b}{b} \leq \frac{2b}{a} \leq \frac{a+b}{b}$, there are infinitely-many values of t such that $\cos t = \frac{2b^2-a^2}{ab}$, and these all correspond to different points on the curve.

Chapter 4

4.1.1 Let U be an open disc in $\mathbb{R}^2$ and $\mathcal{S} = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in U,\ z = 0\}$. If $W = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in U\}$, then W is an open subset of $\mathbb{R}^3$, and $\mathcal{S} \cap W$ is homeomorphic to U by $(x,y,0) \mapsto (x,y)$. So $\mathcal{S}$ is a surface.

4.1.2 The image of $\boldsymbol{\sigma}_\pm^x$ is the intersection of the sphere with the open set $\pm x > 0$ in $\mathbb{R}^3$, and its inverse is the projection $(x,y,z) \mapsto (y,z)$. Similarly for $\boldsymbol{\sigma}_\pm^y$ and $\boldsymbol{\sigma}_\pm^z$. A point of the sphere not in the image of any of the six patches would have to have x, y and z all zero, which is impossible.

4.1.3 Multiplying the two equations gives $(x^2 - z^2)\sin\theta\cos\theta = (1 - y^2)\sin\theta\cos\theta$, so $x^2 + y^2 - z^2 = 1$ unless $\cos\theta = 0$ or $\sin\theta = 0$; if $\cos\theta = 0$, then $x = -z$ and $y = 1$ and if $\sin\theta = 0$ then $x = z$ and $y = -1$, and both of these lines are also contained in the surface. The given line L_θ passes through $(\sin 2\theta, -\cos 2\theta, 0)$ and is parallel to the vector $(\cos 2\theta, \sin 2\theta, 1)$; it follows that we get all of the lines by taking $0 \leq \theta < \pi$.

Let (x, y, z) be a point of the surface; if $x \neq z$, let θ be such that $\cot\theta = (1 - y)/(x - z)$; then (x, y, z) is on L_θ; similarly if $x \neq -z$. The only remaining cases are the points $(0, 0, \pm 1)$, which lie on the lines $L_{\pi/2}$ and L_0. To get a surface patch covering $\mathcal{S}$, define $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ by

$$\boldsymbol{\sigma}(u, v) = (\sin 2\theta, -\cos 2\theta, 0) + t(\cos 2\theta, \sin 2\theta, 1).$$

By the preceding paragraph, this patch covers the whole surface.

Let M_φ be the line $(x-z)\cos\varphi = (1+y)\sin\varphi$, $(x+z)\sin\varphi = (1-y)\cos\varphi$. By the same argument as above, M_φ is contained in the surface and every point of the surface lies on some M_φ with $0 \leq \varphi < \pi$. If $\theta + \varphi$ is not a multiple of π, the lines L_θ and M_φ intersect in the point $(\frac{\cos(\theta-\varphi)}{\sin(\theta+\varphi)}, \frac{\sin(\theta-\varphi)}{\sin(\theta+\varphi)}, \frac{\cos(\theta+\varphi)}{\sin(\theta+\varphi)})$; for each θ with $0 \leq \theta < \pi$, there is exactly one φ with $0 \leq \varphi < \pi$ such that $\theta + \varphi$ is a multiple of π, and the lines L_θ and M_φ do not intersect. If (x, y, z) lies on both L_θ and L_φ, with $\theta \neq \varphi$, then $(1 - y)\tan\theta = (1 - y)\tan\varphi$ and $(1 + y)\cot\theta = (1 - y)\cot\varphi$, which gives both $y = 1$ and $y = -1$ (the case in which $\theta = 0$ and $\varphi = \pi/2$, or vice versa, has to be treated separately, but the conclusion is the same). This shows that L_θ and L_φ do not intersect; similarly, M_θ and M_φ do not intersect.

4.1.4 For the first part, let $U = \{(u, v) \in \mathbb{R}^2 \mid 0 < u^2 + v^2 < \pi^2\}$, let $r = \sqrt{u^2 + v^2}$, and define $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ by $\boldsymbol{\sigma}(u, v) = (\frac{u}{r}, \frac{v}{r}, \tan(r - \frac{\pi}{2}))$.

If S^2 could be covered by a single surface patch $\boldsymbol{\sigma} : U \to \mathbb{R}^3$, then S^2 would be homeomorphic to the open subset U of $\mathbb{R}^2$. As S^2 is a closed and bounded subset of $\mathbb{R}^3$, it is compact. Hence, U would be compact, and hence closed. But, since $\mathbb{R}^2$ is connected, the only non-empty subset of $\mathbb{R}^2$ that is both open and closed is $\mathbb{R}^2$ itself, and this is not compact as it is not bounded.

4.1.5 If $\{\boldsymbol{\sigma}_\alpha : U_\alpha \to \mathbb{R}^3\}$ is an atlas for a surface $\mathcal{S}$, and if W is an open subset of $\mathbb{R}^3$, then the restrictions $\{\boldsymbol{\sigma}|_{U_\alpha \cap \boldsymbol{\sigma}_\alpha^{-1}(W)}\}$ form an atlas of $\mathcal{S} \cap W$ (one should discard the restrictions for which $U_\alpha \cap \boldsymbol{\sigma}_\alpha^{-1}(W)$ is empty).

4.1.6 Parametrize the curve by $\boldsymbol{\gamma}(t) = (\cos\theta, \sin\theta, z)$ where θ and z are smooth functions of t, and assume that $\boldsymbol{\gamma}$ is unit-speed. Then, $\dot{\boldsymbol{\gamma}} = (-\dot{\theta}\sin\theta, \dot{\theta}\cos\theta, \dot{z})$ so $\boldsymbol{\gamma}$ interescts the rulings of the cylinder at a constant angle if and only if $\dot{z}$ is constant, i.e. $z = at + b$ for some constant a, b. Since $\boldsymbol{\gamma}$ is unit-speed, $\dot{\theta}^2 + \dot{z}^2 = 1$ so $\dot{\theta}$ is a constant c such that $a^2 + c^2 = 1$ and $\theta = ct + d$ where d is a constant. Then, $\boldsymbol{\gamma}(t) = (\cos(ct + d), \sin(ct + d), at + b)$. This is a circle if $a = 0$, a straight line if $c = 0$ and a circular helix in all other cases.

4.1.7 Noting that (x, y, z) lies on the ellipsoid if and only if $(\frac{x}{p}, \frac{y}{q}, \frac{z}{q}) \in S^2$, we can use the latitude-longitude parametrization of S^2 to get the parametrization

$$\boldsymbol{\sigma}(\theta, \varphi) = (p\cos\theta\cos\varphi, q\cos\theta\sin\varphi, r\sin\theta)$$

of the ellipsoid. $\boldsymbol{\sigma}$ is continuous and is a homeomorphism onto its image if (θ, φ) is restricted in the same way as for S^2 (Example 4.1.4).

4.1.8 If $x = \sin u$, $y = \sin v$ and $z = \sin(u + v)$, then

$$\begin{aligned} x^2 - y^2 + z^2 &= \sin^2 u - \sin^2 v + \sin^2(u + v) \\ &= \sin^2 u - \sin^2 v + \sin^2 u\cos^2 v + \cos^2 u\sin^2 v + 2\sin u\sin v\cos u\cos v \\ &= \sin^2 u + \sin^2 u\cos^2 v - \sin^2 v(1 - \cos^2 u) + 2\sin u\sin v\cos u\cos v \\ &= (1 - \sin^2 v)\sin^2 u + \sin^2 u\cos^2 v + 2\sin u\sin v\cos u\cos v \\ &= 2\sin^2 u\cos^2 v + 2\sin u\sin v\cos u\cos v \\ &= 2\sin u\cos v(\sin u\cos v + \cos u\sin v) \\ &= 2\sin u\cos v\sin(u + v). \end{aligned}$$

Hence, $(x^2 - y^2 + z^2)^2 = 4\sin^2 u(1 - \sin^2 v)\sin^2(u + v) = 4x^2(1 - y^2)z^2$. Now, $\boldsymbol{\sigma}$ is clearly continuous, as is its inverse map $(x, y, z) \mapsto (\sin^{-1} x, \sin^{-1} y)$, so $\boldsymbol{\sigma}$ is a homeomorphism.

4.2.1 $\boldsymbol{\sigma}$ is obviously smooth and $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (-f_u, -f_v, 1)$ is nowhere zero, so $\boldsymbol{\sigma}$ is regular.

4.2.2 $\boldsymbol{\sigma}^z_\pm$ is a special case of Exercise 4.2.1, with $f = \pm\sqrt{1 - u^2 - v^2}$ ($\sqrt{1 - u^2 - v^2}$ is smooth because $1 - u^2 - v^2 > 0$ if $(u, v) \in U$); similarly for the other patches. The transition map from $\boldsymbol{\sigma}^x_+$ to $\boldsymbol{\sigma}^y_+$, for example, is $\Phi(\tilde{u}, \tilde{v}) = (u, v)$, where $\boldsymbol{\sigma}^y_+(\tilde{u}, \tilde{v}) = \boldsymbol{\sigma}^x_+(u, v)$; so $u = \sqrt{1 - \tilde{u}^2 - \tilde{v}^2}$, $v = \tilde{v}$, and this is smooth since $1 - \tilde{u}^2 - \tilde{v}^2 > 0$ if $(\tilde{u}, \tilde{v}) \in U$.

4.2.3 (i) is clearly injective and is regular because $\boldsymbol{\sigma}$ is smooth and $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (-v, -u, 1)$ is never zero.
(ii) is injective but is not regular since $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (0, -3v^2, 2v)$ vanishes when $v = 0$.
(iii) is not injective because $\boldsymbol{\sigma}(u, v) = (\boldsymbol{\sigma}(-u - 1, v)$ and is also not regular since $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (0, 2v(1 + 2u), 1 + 2u)$ vanishes when $u = -1/2$.

4.2.4 This is similar to Example 4.1.4, but using the 'latitude-longitude' patch $\boldsymbol{\sigma}(\theta, \varphi) = (p\cos\theta\cos\varphi, q\cos\theta\sin\varphi, r\sin\theta)$.

4.2.5 A typical point on the circle $\mathcal{C}$ has coordinates $(a + b\cos\theta, 0, b\sin\theta)$; rotating this about the z-axis through an angle φ gives the point $\boldsymbol{\sigma}(\theta, \varphi)$; the torus is covered

by the four patches obtained by taking (θ, φ) to lie in one of the following open sets:
(i) $0 < \theta < 2\pi, 0 < \varphi < 2\pi$;
(ii) $0 < \theta < 2\pi, -\pi < \varphi < \pi$;
(iii) $-\pi < \theta < \pi, 0 < \varphi < 2\pi$;
(iv) $-\pi < \theta < \pi, -\pi < \varphi < \pi$.
Each patch is regular because

$$\boldsymbol{\sigma}_\theta \times \boldsymbol{\sigma}_\varphi = -b(a + b\cos\theta)(\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$$

is never zero (since $\| \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \| = b(a + b\cos\theta) \geq b(a - b) > 0$).

4.2.6 Suppose the centre of the propeller is initially at the origin. At time t, the centre is at $(0, 0, \alpha t)$ where α is the speed of the aeroplane. If the propeller is initially along the x-axis, the point initially at $(v, 0, 0)$ is therefore at the point $(v\cos\omega t, v\sin\omega t, \alpha t)$ at time t, where ω is the angular velocity of the propeller. Let $u = \omega t$, $\lambda = \alpha/\omega$. Next, $\boldsymbol{\sigma}_u = (-v\sin u, v\cos u, \lambda)$, $\boldsymbol{\sigma}_v = (\cos u, \sin u, 0)$, so the standard unit normal is $\mathbf{N} = (\lambda^2 + v^2)^{-1/2}(-\lambda\sin u, \lambda\cos u, -v)$. If θ is the angle between $\mathbf{N}$ and the z-axis, $\cos\theta = -v/(\lambda^2 + v^2)^{1/2}$ and hence $\cot\theta = \pm v/\lambda$, while the distance from the z-axis is v.

4.2.7 $\boldsymbol{\sigma}$ is the tube swept out by a circle of radius a in a plane perpendicular to $\boldsymbol{\gamma}$ as its centre moves along $\boldsymbol{\gamma}$. $\boldsymbol{\sigma}_s = (1 - \kappa a\cos\theta)\mathbf{t} - \tau a\sin\theta\mathbf{n} + \tau a\cos\theta\mathbf{b}$, $\boldsymbol{\sigma}_\theta = -a\sin\theta\mathbf{n} + a\cos\theta\mathbf{b}$, giving $\boldsymbol{\sigma}_s \times \boldsymbol{\sigma}_\theta = -a(1 - \kappa a\cos\theta)(\cos\theta\mathbf{n} + \sin\theta\mathbf{b})$; this is never zero since $\kappa a < 1$ implies that $1 - \kappa a\cos\theta > 0$ for all θ. The first fundamental form is $((1 - \kappa a\cos\theta)^2 + \tau^2 a^2)\, ds^2 + 2\tau a^2\, ds d\theta + a^2\, d\theta^2$, so the area is $\int_{s_0}^{s_1}\int_0^{2\pi} a(1 - \kappa a\cos\theta) ds d\theta = 2\pi a(s_1 - s_0)$.

4.2.8 If $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \mathbf{a}$, where $\mathbf{a}$ is a constant vector, then $\tilde{\boldsymbol{\sigma}}$ is smooth if $\boldsymbol{\sigma}$ is smooth, and $\tilde{\boldsymbol{\sigma}}_u = \boldsymbol{\sigma}_u$, $\tilde{\boldsymbol{\sigma}}_v = \boldsymbol{\sigma}_v$, so $\tilde{\boldsymbol{\sigma}}$ is regular if $\boldsymbol{\sigma}$ is regular. If A is an invertible 3×3 matrix and $\tilde{\boldsymbol{\sigma}} = A\boldsymbol{\sigma}$, then $\tilde{\boldsymbol{\sigma}}$ is smooth if $\boldsymbol{\sigma}$ is smooth and $\tilde{\boldsymbol{\sigma}}_u = A\boldsymbol{\sigma}_u$, $\tilde{\boldsymbol{\sigma}}_v = A\boldsymbol{\sigma}_v$, so since A is invertible $\tilde{\boldsymbol{\sigma}}_u$ and $\tilde{\boldsymbol{\sigma}}_v$ are linearly independent if $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are linearly independent.

4.2.9 See Exercise 4.1.5. The restriction of a smooth map $U \to \mathbb{R}^2$, where U is an open subset of $\mathbb{R}^2$, to an open subset of U is smooth.

4.2.10 The map $\boldsymbol{\sigma}$ is a diffeomorphism, since it is smooth, obviously bijective, and has smooth inverse $(x, y, z) \mapsto (x, y)$.

4.2.11 We find that $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (-\cos v\cos(u + v), -\cos u\cos(u + v), \cos u\cos v)$. This is non-zero when $-\pi/2 < u, v < \pi/2$ since $\cos u$ and $\cos v$ are non-zero in this range.

4.2.12 The torus is covered by the patches $\boldsymbol{\sigma} : U_n \to \mathbb{R}^3$, $n = 1, 2, 3$, where $\boldsymbol{\sigma}$ is as in

the question and

$$\begin{aligned}
U_1 &= \{(\theta,\varphi)\,|\,0<\theta<2\pi, 0<\varphi<2\pi\},\\
U_2 &= \left\{(\theta,\varphi)\,|\,-\frac{2\pi}{3}<\theta<\frac{4\pi}{3}, -\frac{4\pi}{3}<\varphi<\frac{2\pi}{3}\right\},\\
U_3 &= \left\{(\theta,\varphi)\,|\,-\frac{4\pi}{3}<\theta<\frac{2\pi}{3}, -\frac{2\pi}{3}<\varphi<\frac{4\pi}{3}\right\}.
\end{aligned}$$

Suppose for a contradiction that the torus is covered by two patches $\boldsymbol{\sigma} : V \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : V' \to \mathbb{R}^3$, where $\boldsymbol{\sigma}$ is as before and V, V' are open subsets of $\mathbb{R}^2$. We might as well assume that

$$\begin{aligned}
V &= \{(\theta,\varphi)\,|\,\alpha<\theta<\alpha+2\pi, \beta<\varphi<\beta+2\pi\},\\
V' &= \{(\theta,\varphi)\,|\,\alpha'<\theta<\alpha'+2\pi, \beta'<\varphi<\beta'+2\pi\},
\end{aligned}$$

for some $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$. But neither of the patches $\boldsymbol{\sigma} : V \to \mathbb{R}^3$ or $\boldsymbol{\sigma} : V' \to \mathbb{R}^3$ contains the point $\boldsymbol{\sigma}(\alpha', \beta)$ (for example).

4.2.13 Taking $f = 3xy - z(z+4) + c$, we have $\nabla f = (3y, 3x, -2z-4)$ which is zero only at the point $(0,0,-2)$. But $f(0,0,-2) = 0$ only if $c = -4$. So if $c \neq -4$, $f = 0$ is a smooth surface.

4.2.14 Taking $f = x^3 + 3(y^2+z^2)^2 - 2$, we have $\nabla f = (3x^2, 12y(y^2+z^2), 12z(y^2+z^2))$ which is zero only at the origin. But $f(0,0,0) \neq 0$, so $f = 0$ is a smooth surface.

4.2.15 Take $f = x^{2/3} + y^{2/3} + z^{2/3} = 1$. Not that f is smooth provided x, y and z are never zero, i.e. if we exclude the points on the coordinate planes. Then $\nabla f = (\frac{2}{3}x^{-1/3}, \frac{2}{3}y^{-1/3}, \frac{2}{3}z^{-1/3})$ which is non-zero.

4.2.16 If $xyz = 1$ then x, y and z are all non-zero. By the intermediate value theorem, a smooth (or even just continuous) curve on the surface cannot connect a point with $x > 0$ (for example) to a point with $x < 0$ (as it would have to pass through a point with $x = 0$). Thus, two points in the same connected piece of the surface must have x-coordinates of the same sign (and similarly for their y- and z-coordinates). Note that the sign of any two of the coordinates determines that of the third, since $xyz = 1$.

Consider the piece with x, y and z all > 0. This can be parametrized using the single patch $\boldsymbol{\sigma}(u,v) = (u, v, 1/uv)$, defined on the open set $\{(u,v)\,|\,u > 0, v > 0\}$. It is clear that $\boldsymbol{\sigma}$ is smooth and injective.

4.2.17 Consider the circular helix $\boldsymbol{\gamma}(\theta) = (a\cos\theta, a\sin\theta, b\theta)$. The mid-point of the chord joining $\boldsymbol{\gamma}(\theta)$ and $\boldsymbol{\gamma}(\varphi)$ is the point

$$\left(\frac{1}{2}a(\cos\theta+\cos\varphi), \frac{1}{2}a(\sin\theta+\sin\varphi), \frac{1}{2}b(\theta+\varphi)\right) = (v\cos u, v\sin u, bu),$$

where $u = \frac{1}{2}(\theta + \varphi)$, $v = a\cos\frac{1}{2}(\theta - \varphi)$. Hence, the set of mid-points of the chords of the helix is the subset of the helicoid in Exercise 4.2.6 with $\lambda = b$ given by $|v| \leq |a|$.

4.3.1 If $\mathcal{S}$ is covered by a single surface patch $\boldsymbol{\sigma} : U \to \mathbb{R}^3$, then $f : \mathcal{S} \to \mathbb{R}$ is smooth if and only if $f \circ \boldsymbol{\sigma} : U \to \mathbb{R}$ is smooth. We must check that, if $\tilde{\boldsymbol{\sigma}} : \tilde{U} \to \mathbb{R}^3$ is another patch covering $\mathcal{S}$, then $f \circ \tilde{\boldsymbol{\sigma}}$ is smooth if and only if $f \circ \boldsymbol{\sigma}$ is smooth. This is true because $f \circ \tilde{\boldsymbol{\sigma}} = (f \circ \boldsymbol{\sigma}) \circ \Phi$, where Φ is the transition map from $\boldsymbol{\sigma}$ to $\tilde{\boldsymbol{\sigma}}$, and both Φ and Φ^{-1} are smooth. The last part is true because if $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ is a smooth map, where U is an open subset of $\mathbb{R}^2$, then each component of $\boldsymbol{\sigma}$ (which is a map $U \to \mathbb{R}$) is smooth (this is because a vector function such as $\boldsymbol{\sigma}$ is differentiated 'componentwise').

4.3.2 f is not a diffeomorphism as it is not injective: $f(0, y, z) = f(0, y, z + 2\pi)$. Take an atlas for the cone consisting of the patches $\boldsymbol{\sigma}(u, v) = (u\cos v, u\sin v, u)$, defined on the open sets

$$U_1 = \{(u, v)|u > 0, 0 < v < 2\pi\} \text{ and } U_2 = \{(u, v)|u > 0, -\pi < v < \pi\}$$

(call these $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$), and parametrize the half-plane by $\boldsymbol{\pi}(u, v) = (0, u, v)$ with $u > 0$. If $(0, a, b)$ is any point in the plane, assume first that b is not an even multiple of π, say $2n\pi < b < 2(n + 1)\pi$ for some integer n. Then, $f(\boldsymbol{\pi}(u, v)) = \boldsymbol{\sigma}_1(u, v - 2n\pi)$ if $2n\pi < v < 2(n + 1)\pi$. So f is a diffeomorphism from the open subset $\{(0, y, z)|2n\pi < z < 2(n + 1)\pi\}$ of the half-plane to the cone with the half-line $y = 0, x = z > 0$ removed. Similarly if b is not an odd multiple of π. This proves that f is a local diffeomorphism.

4.4.1 (i) At $(1, 1, 0)$, $\boldsymbol{\sigma}_u = (1, 0, 2)$, $\boldsymbol{\sigma}_v = (0, 1, -2)$ so $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (-2, 2, 1)$ and the tangent plane is $-2x + 2y + z = 0$.
(ii) At $(1, 0, 1)$, where $r = 1, \theta = 0$, $\boldsymbol{\sigma}_r = (1, 0, 2), \boldsymbol{\sigma}_\theta = (0, 1, 2)$ so $\boldsymbol{\sigma}_r \times \boldsymbol{\sigma}_\theta = (-2, -2, 1)$ and the equation of the tangent plane is $-2x - 2y + z = 0$.

4.4.2 Let $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ be a reparametrization of $\boldsymbol{\sigma}$. Then,

$$\boldsymbol{\sigma}_u = \frac{\partial \tilde{u}}{\partial u}\tilde{\boldsymbol{\sigma}}_{\tilde{u}} + \frac{\partial \tilde{v}}{\partial u}\tilde{\boldsymbol{\sigma}}_{\tilde{v}}, \ \boldsymbol{\sigma}_v = \frac{\partial \tilde{u}}{\partial v}\tilde{\boldsymbol{\sigma}}_{\tilde{u}} + \frac{\partial \tilde{v}}{\partial v}\tilde{\boldsymbol{\sigma}}_{\tilde{v}},$$

so $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are linear combinations of $\tilde{\boldsymbol{\sigma}}_{\tilde{u}}$ and $\tilde{\boldsymbol{\sigma}}_{\tilde{v}}$. Hence, any linear combination of $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ is a linear combination of $\tilde{\boldsymbol{\sigma}}_{\tilde{u}}$ and $\tilde{\boldsymbol{\sigma}}_{\tilde{v}}$. The converse is also true since $\boldsymbol{\sigma}$ is a reparametrization of $\tilde{\boldsymbol{\sigma}}$.

4.4.3 If $\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$ then $\frac{d}{dt}F(\boldsymbol{\gamma}(t)) = F_x\dot{x} + F_y\dot{y} + F_z\dot{z} = \boldsymbol{\nabla}F.\dot{\boldsymbol{\gamma}}$. Since $\boldsymbol{\nabla}_{\mathcal{S}}F - \boldsymbol{\nabla}F$ is perpendicular to $T_{\mathbf{p}}\mathcal{S}$, it is perpendicular to $\dot{\boldsymbol{\gamma}}(t_0)$ for every curve $\boldsymbol{\gamma}$ on $\mathcal{S}$ passing through $\mathbf{p}$ when $t = t_0$. It follows that $\boldsymbol{\nabla}_{\mathcal{S}}F.\dot{\boldsymbol{\gamma}} = \boldsymbol{\nabla}F.\dot{\boldsymbol{\gamma}}$ at $\mathbf{p}$. If the restriction of F to $\mathcal{S}$ has a local maximum or a local minimum at $\mathbf{p}$, so does

$F(\boldsymbol{\gamma}(t))$ for all curves $\boldsymbol{\gamma}$ on $\mathcal{S}$ passing through $\mathbf{p}$, hence $\frac{d}{dt}F(\boldsymbol{\gamma}(t)) = 0$ at $\mathbf{p}$, which implies that $\boldsymbol{\nabla} F$ is perpendicular to $\dot{\boldsymbol{\gamma}}$, and hence perpendicular to the tangent plane of $\mathcal{S}$ at $\mathbf{p}$. This means that $\boldsymbol{\nabla}_{\mathcal{S}} F = \mathbf{0}$.

4.4.4 $d(f \circ \boldsymbol{\gamma})/dt) = D_{\boldsymbol{\gamma}(t)} f(\dot{\boldsymbol{\gamma}}(t))$ is non-zero because $\dot{\boldsymbol{\gamma}}$ is non-zero ($\boldsymbol{\gamma}$ is regular) and $D_{\boldsymbol{\gamma}(t)} f$ is invertible (Proposition 4.4.6).

4.4.5 We find that, at $\theta = \varphi = \pi/4$, $\boldsymbol{\sigma}_\theta = \frac{1}{2}b(-1,-1,\sqrt{2})$, $\boldsymbol{\sigma}_\varphi = \frac{1}{2}(a\sqrt{2}+b)(-1,1,0)$, so the unit normal of the torus at this point is parallel to

$$(-1,-1,\sqrt{2}) \times (-1,1,0) = -\sqrt{2}(1,1,\sqrt{2}).$$

Hence, the tangent plane is $x + y + \sqrt{2}z = 0$.

4.5.1 The transition map $\Phi(t,\theta) = (\tilde{t},\tilde{\theta})$ is defined on the union of the rectangles given by $0 < \theta < \pi$ and $\pi < \theta < 2\pi$ (and $-1/2 < t < 1/2$). Obviously $\Phi(t,\theta) = (t,\theta)$ if $0 < \theta < \pi$. If $\pi < \theta < 2\pi$, we must have $\tilde{\theta} = \theta - 2\pi$. Since $\sin\frac{\tilde{\theta}}{2} = -\sin\frac{\theta}{2}$, $\cos\frac{\tilde{\theta}}{2} = -\cos\frac{\theta}{2}$, $\boldsymbol{\sigma}(t,\theta) = \tilde{\boldsymbol{\sigma}}(\tilde{t},\tilde{\theta})$ forces $\tilde{t} = -t$. So $\Phi(t,\theta) = (t,\theta)$ if $0 < \theta < \pi$, and $= (-t, \theta - 2\pi)$ if $\pi < \theta < 2\pi$. The Jacobian determinant is $+1$ on the first rectangle, -1 on the second.

4.5.2 Let $\{\boldsymbol{\sigma}_\alpha : U_\alpha \to \mathbb{R}^3\}$ be an atlas for $\mathcal{S}$ such that the transition map $\Phi_{\alpha\beta}$ between $\boldsymbol{\sigma}_\alpha$ and $\boldsymbol{\sigma}_\beta$ satisfies $\det(J(\Phi_{\alpha\beta})) > 0$ for all α, β (Definition 4.5.1). By Proposition 4.3.1, $\{f \circ \boldsymbol{\sigma}_\alpha\}$ is an atlas for $\tilde{\mathcal{S}}$, and the transition maps for this atlas are the same as those for the atlas of $\mathcal{S}$, because $(f \circ \boldsymbol{\sigma}_\beta)^{-1} \circ (f \circ \boldsymbol{\sigma}_\alpha) = \boldsymbol{\sigma}_\beta^{-1} \circ \boldsymbol{\sigma}_\alpha$ (where this composite is defined). So the atlas $\{f \circ \boldsymbol{\sigma}_\alpha\}$ gives $\tilde{\mathcal{S}}$ the structure of an oriented surface.

4.5.3 If $\boldsymbol{\sigma}(\theta,\varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$, we find that $\boldsymbol{\sigma}_\theta \times \boldsymbol{\sigma}_\varphi = -\cos\theta\boldsymbol{\sigma}(\theta,\varphi)$. Hence, the standard unit normal points inwards if $\cos\theta > 0$ (if $-\pi/2 < \theta < \pi/2$, for example) and outwards of $\cos\theta < 0$. Note that the points at which $\cos\theta = 0$ (i.e. the north and south pole) must be excluded for $\boldsymbol{\sigma}$ to be regular.

For the patches $\boldsymbol{\sigma}_\pm^x$, for example, we find that $(\boldsymbol{\sigma}_\pm^x)_u \times (\boldsymbol{\sigma}_\pm^x)_v = \pm\frac{1}{\sqrt{1-u^2-v^2}}\boldsymbol{\sigma}_\pm^x$, so the standard unit normal of $\boldsymbol{\sigma}_+^x$ points outwards and that of $\boldsymbol{\sigma}_-^x$ points inwards.

4.5.4 Apply the proof of Proposition 2.2.1 with $f = \dot{\boldsymbol{\gamma}}.\mathbf{v}$, $g = \dot{\boldsymbol{\gamma}}.\tilde{\mathbf{v}}$.

4.5.5 From $\boldsymbol{\sigma}^* = \frac{\boldsymbol{\sigma}}{\boldsymbol{\sigma}.\boldsymbol{\sigma}}$, we get

$$\boldsymbol{\sigma}_u^* = \frac{(\boldsymbol{\sigma}.\boldsymbol{\sigma})\boldsymbol{\sigma}_u - 2(\boldsymbol{\sigma}_u.\boldsymbol{\sigma})\boldsymbol{\sigma}}{\parallel \boldsymbol{\sigma} \parallel^4},$$

with a similar formula for $\boldsymbol{\sigma}_v^*$. This gives

$$\begin{aligned}
\boldsymbol{\sigma}_u^* \times \boldsymbol{\sigma}_v^* &= \frac{1}{\| \boldsymbol{\sigma} \|^6} \big(\| \boldsymbol{\sigma} \|^2 (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) - 2\boldsymbol{\sigma} \times ((\boldsymbol{\sigma}_u.\boldsymbol{\sigma})\boldsymbol{\sigma}_v - (\boldsymbol{\sigma}_v.\boldsymbol{\sigma})\boldsymbol{\sigma}_u)\big) \\
&= \frac{1}{\| \boldsymbol{\sigma} \|^6} \big(\| \boldsymbol{\sigma} \|^2 (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) + 2\boldsymbol{\sigma} \times (\boldsymbol{\sigma} \times (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v))\big) \\
&= \frac{1}{\| \boldsymbol{\sigma} \|^6} \big(2(\boldsymbol{\sigma}.(\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v))\boldsymbol{\sigma} - \| \boldsymbol{\sigma} \|^2 (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v)\big) \\
&= \frac{\| \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \|}{\| \boldsymbol{\sigma} \|^4} \left(\frac{2(\boldsymbol{\sigma}.\mathbf{N})\boldsymbol{\sigma}}{\| \boldsymbol{\sigma} \|^2} - \mathbf{N}\right).
\end{aligned}$$

Hence, it suffices to show that $\frac{2(\boldsymbol{\sigma}.\mathbf{N})\boldsymbol{\sigma}}{\|\boldsymbol{\sigma}\|^2} - \mathbf{N}$ is a unit vector. But its squared length is

$$\frac{4(\boldsymbol{\sigma}.\mathbf{N})^2}{\| \boldsymbol{\sigma} \|^2} - \frac{4(\boldsymbol{\sigma}.\mathbf{N})^2}{\| \boldsymbol{\sigma} \|^2} + \| \mathbf{N} \|^2 = 1.$$

Chapter 5

5.1.1 (i) $f_x = 2x$, $f_y = 2y$ and $f_z = 4z^3$ vanish simultaneously only when $x = y = z = 0$, but this does not satisfy $x^2 + y^2 + z^4 = 1$. So by Theorem 5.1.1 this is a smooth surface.

(ii) Let $f(x, y, z)$ be the left-hand side minus the right-hand side; then,

$$\begin{aligned}
f_x &= 4x(x^2 + y^2 + z^2 - a^2 - b^2), \\
f_y &= 4y(x^2 + y^2 + z^2 - a^2 - b^2), \\
f_z &= 4z(x^2 + y^2 + z^2 + a^2 - b^2);
\end{aligned}$$

if $f_z = 0$ then $z = 0$ since $x^2 + y^2 + z^2 + a^2 - b^2 > 0$ everywhere on the torus; if $f_x = f_y = 0$ too, then since the origin is not on the torus, we must have $x^2 + y^2 = a^2 + b^2$, but then substituting into the equation of the torus gives $(2a^2)^2 = 4a^2(a^2 + b^2)$, a contradiction.

For the last part, let $\boldsymbol{\sigma}(\theta, \varphi) = (x, y, z)$ be the parametrization in Exercise 4.2.5. Then, $x^2 + y^2 + z^2 + a^2 - b^2 = 2a(a + b\cos\theta)$, so

$$(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(a + b\cos\theta)^2 = 4a^2(x^2 + y^2).$$

Conversely, if (x, y, z) satisfies the given equation, let $r = \sqrt{x^2 + y^2}$. A little algebra gives $(r^2 + z^2 - a^2 - b^2)^2 = 4a^2(b^2 - z^2)$. Hence, $|z| \le b$ so $z = b\sin\theta$ for some $\theta \in \mathbb{R}$. Then we find $r^2 = a^2 + b^2\cos^2\theta \pm 2ab\cos\theta$, so (since $r \ge 0$) $r = a \pm b\cos\theta$. With the plus sign $(x, y, z) = \boldsymbol{\sigma}(\theta, \varphi)$ for some $\varphi \in \mathbb{R}$; with the

minus sign, $(x, y, z) = \boldsymbol{\sigma}(\pi - \theta, \varphi)$ for some φ. Thus, the image of $\boldsymbol{\sigma} : \mathbb{R}^2 \to \mathbb{R}^3$ coincides with the set of solutions to the given equation.

5.1.2 See the solution of Exercise 4.4.3 for the first part. Since $\mathcal{S}$ has a (smooth) choice of unit normal $\boldsymbol{\nabla} f \parallel \boldsymbol{\nabla} f \parallel$ at each point, it is orientable. The solution of Exercise 4.4.3 also shows that if the restriction of F to $\mathcal{S}$ has a local maximum or a local minimum at $\mathbf{p}$, then $\boldsymbol{\nabla} F$ is perpendicular to the tangent plane of $\mathcal{S}$ at $\mathbf{p}$. But $\boldsymbol{\nabla} f$ is also perpendicular to the tangent plane. Hence, if the restriction of F to $\mathcal{S}$ has a local maximum or a local minimum at $\mathbf{p}$, then $\boldsymbol{\nabla} F$ is parallel to $\boldsymbol{\nabla} f$ at $\mathbf{p}$, i.e. $\boldsymbol{\nabla} F = \lambda \boldsymbol{\nabla} f$ for some scalar λ.

5.1.3 Let $f(x, y, z) = xyz - 1$, $F(x, y, z) = x^2 + y^2 + z^2$. Then, $f = 0$ is a smooth surface $\mathcal{S}$ by Theorem 5.1.1 and F defines a smooth function on $\mathcal{S}$. To see that F has a smallest value on $\mathcal{S}$, let $\mathcal{B}$ be the closed ball given by $x^2 + y^2 + z^2 \leq 3$. Then, $\mathcal{B} \cap \mathcal{S}$ is compact as it is closed and bounded and it is non-empty because it contains the point $(1, 1, 1)$. Hence, the continuous positive function F must attain its lower bound, say ℓ, on $\mathcal{B} \cap \mathcal{S}$, and $\ell \leq 3$ since $F(1, 1, 1) = 3$. Obviously $F(x, y, z) > 3$ if $(x, y, z) \notin \mathcal{B}$, so ℓ is the smallest value of F on $\mathcal{S}$.

By Exercise 5.1.2, the local maxima or minima of F on $\mathcal{S}$ occur where $(2x, 2y, 2z)$ $= \lambda(yz, xz, xy)$ for some λ. Since $xyz = 1$ on $\mathcal{S}$ this gives $x^2 = y^2 = z^2 = \lambda/2$, so x, y, z are equal up to sign. Since their product is 1, there are four possibilities: $(x, y, z) = (1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$ or $(-1, -1, 1)$. The value of F is 3 at each of these points, which is the smallest value of F on $\mathcal{S}$ from above. The distance between any two of these points of $\mathbb{R}^3$ is the same $(2\sqrt{2})$, so they form the vertices of a regular tetrahedron.

5.2.1 (i) $(p \cos u \cos v, q \cos u \sin v, r \sin u)$ (cf. Exercise 4.2.4); (ii) see Exercise 4.1.3; (iii) $(u, v, \pm\sqrt{1 + \frac{u^2}{p^2} + \frac{v^2}{q^2}})$; (iv), (v), (vi) see Exercise 4.2.1; (vii) $(p \cos u, q \cos u, v)$; (viii) $(\pm p \cosh u, q \sinh u, v)$; (ix) $(u, u^2/p^2, v)$; (x) $(0, u, v)$; (xi) $(\pm p, u, v)$.

5.2.2 In the notation of Theorem 5.2.2, $A = \begin{pmatrix} 1 & -1/3 & 0 \\ -1/3 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. The eigenvalues are $2/3, 4/3, -2$ and the corresponding unit eigenvectors are the columns of $P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. If $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = P \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then $z' = z$ and the quadric becomes $\frac{2}{3}{x'}^2 + \frac{4}{3}{y'}^2 - 2{z'}^2 + 4z' = c$, i.e. $\frac{2}{3}{x'}^2 + \frac{4}{3}{y'}^2 - 2(z' - 1)^2 = c - 2$. Comparing with the standard forms in Theorem 5.2.2 gives the stated results when $c > 2$ and $c < 2$. If $c = 2$ we have a cone with axis the z-axis (which is the same as the z'-axis), vertex at $x' = y' = 0, z' = 1$, i.e. $x = y = 0, z = 1$, and cross-section perpendicular to the z-axis an ellipse $\frac{2}{3}{x'}^2 + \frac{4}{3}{y'}^2 = \text{constant}$.

5.2.3 Substituting the components (x, y, z) of $\boldsymbol{\gamma}(t) = \mathbf{a} + t\mathbf{b}$ into the equation of the quadric gives a quadratic equation for t; if the quadric contains three points on the line, this quadratic equation has three roots, hence is identically zero, so the quadric contains the whole line.

For the second part, take three points on each of the given lines; substituting the coordinates of these nine points into the equation of the quadric gives a system of nine homogeneous linear equations for the ten coefficients $a_1, \ldots, c$ of the quadric; such a system always has a non-trivial solution. By the first part, the resulting quadric contains all three lines.

5.2.4 Let L_1, L_2, L_3 be three lines from the first family; by the preceding exercise, there is a quadric $\mathcal{Q}$ containing all three lines; all but finitely-many lines of the second family intersect each of the three lines; if L' is such a line, $\mathcal{Q}$ contains three points of L', and hence the whole of L' by the preceding exercise; so $\mathcal{Q}$ contains all but finitely-many lines of the second family; since any quadric is a closed subset of $\mathbb{R}^3$, $\mathcal{Q}$ must contain all the lines of the second family, and hence must contain $\mathcal{S}$.

5.2.5 $z = \left(\frac{x}{p} - \frac{y}{q}\right)\left(\frac{x}{p} + \frac{y}{q}\right) = uv$, $x = \frac{1}{2}p(u+v)$, $y = \frac{1}{2}q(v-u)$, so a parametrisation is $\boldsymbol{\sigma}(u, v) = (\frac{1}{2}p(u+v), \frac{1}{2}q(v-u), uv)$; $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (-\frac{1}{2}q(u+v), \frac{1}{2}p(v-u), pq)$ is never zero so $\boldsymbol{\sigma}$ is regular. For a fixed value of u, $\boldsymbol{\sigma}(u, v) = (\frac{1}{2}pu, -\frac{1}{2}qu, 0) + v(\frac{1}{2}p, \frac{1}{2}q, u)$ is a straight line; similarly for a fixed value of v; hence the hyperbolic paraboloid is the union of each of two families of straight lines.

5.2.6 Let $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$. Since A is a real symmetric matrix, there is an orthogonal matrix P such that $PAP^t = A'$ is a diagonal matrix, say $A' = \begin{pmatrix} a' & 0 \\ 0 & b' \end{pmatrix}$ (see Appendix 0). We can assume that $\det(P) = 1$ by changing the sign of one of the rows of P if necessary. Then the equation of the conic becomes

$$a'x'^2 + b'y'^2 + d'x' + e'y' + f = 0,$$

where $\begin{pmatrix} x' \\ y' \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix}$, $\begin{pmatrix} d' \\ e' \end{pmatrix} = P\begin{pmatrix} d \\ e \end{pmatrix}$. In other words, we might as well assume that $c = 0$ from the beginning.

Assume then that $c = 0$. If $a \neq 0$, we can assume that $d = 0$ by replacing x by $x + \frac{d}{a}$ (a translation). Similarly, if $b \neq 0$ we can assume that $e = 0$.

If a and b are both $\neq 0$, we can therefore assume that $d = e = 0$ and we are in case (i) or (iii).

If $a = 0$ and $b \neq 0$, the equation takes the form $by^2 + dx + f = 0$. If $d \neq 0$ replacing x by $x + \frac{f}{d}$ we can assume that $f = 0$ and then we are in case (ii). If $d = 0$ we are in case (v). The case $a \neq 0, b = 0$ is similar.

If $a = b = 0$ we have a straight line.

5.2.7 (i) From the classification in Theorem 5.2.2, the connected quadric surfaces are those of types (i), (ii), (iv), (v), (vii), (ix) and (x) (remember that for type (vi) to be a smooth surface, we have to remove the vertex which results in a disconnected surface). Type (i) is diffeomorphic to S^2 by $(x, y, z) \mapsto (\frac{x}{p}, \frac{y}{q}, \frac{z}{r})$; type (ii) to the unit cylinder by $(x, y, z) \mapsto \left(\frac{\frac{x}{p}}{\sqrt{\frac{x^2}{p^2}+\frac{y^2}{q^2}}}, \frac{\frac{y}{q}}{\sqrt{\frac{x^2}{p^2}+\frac{y^2}{q^2}}}, z \right)$; types (iv) and (v) to a plane by $(x, y, z) \mapsto (x, y, 0)$; type (vii) to the unit cylinder by $(x, y, z) \mapsto (\frac{x}{p}, \frac{y}{q}, z)$; type (ix) to a plane by $(x, y, z) \mapsto (x, 0, z)$; and type (x) is a plane.
(ii) For types (iii) and (vi) the connected pieces are the parts of the quadric with $z > 0$ and with $z < 0$. Each part of type (iii) is diffeomorphic to the plane by $(x, y, z) \mapsto (x, y, 0)$. Each part of type (vi) is diffeomorphic to the unit cylinder by $(x, y, z) \mapsto \left(\frac{\frac{x}{p}}{\sqrt{\frac{x^2}{p^2}+\frac{y^2}{q^2}}}, \frac{\frac{y}{q}}{\sqrt{\frac{x^2}{p^2}+\frac{y^2}{q^2}}}, \ln|z| \right)$. The two connected pieces of type (viii) are given by $x > 0$ and $x < 0$; each piece is diffeomorphic to the plane by $(x, y, z) \mapsto (0, y, z)$.

5.3.1 From Example 5.3.2, the surface can be parametrized by

$$\boldsymbol{\sigma}(u, v) = (\cosh u \cos v, \cosh u \sin v, u),$$

with $u \in \mathbb{R}$ and $-\pi < v < \pi$ or $0 < v < 2\pi$.

5.3.2 $\parallel \boldsymbol{\sigma}(u, v) \parallel^2 = \operatorname{sech}^2 u(\cos^2 v + \sin^2 v) + \tanh^2 u = \operatorname{sech}^2 u + \tanh^2 u = 1$, so $\boldsymbol{\sigma}$ parametrizes an open subset of S^2; $\boldsymbol{\sigma}$ is clearly smooth; and $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = -\operatorname{sech}^2 u\, \boldsymbol{\sigma}(u, v)$ is never zero, so $\boldsymbol{\sigma}$ is regular. Meridians correspond to the parameter curves $v =$ constant, and parallels to the curves $u =$ constant.

5.3.3 (i) $\tilde{\boldsymbol{\gamma}}.\mathbf{a} = 0$ so $\tilde{\boldsymbol{\gamma}}$ is contained in the plane perpendicular to $\mathbf{a}$ and passing through the origin;
(ii) simple algebra;
(iii) $\tilde{v}$ is clearly a smooth function of (u, v) and the Jacobian matrix of the map $(u, v) \mapsto (u, \tilde{v})$ is $\begin{pmatrix} 1 & 0 \\ \dot{\boldsymbol{\gamma}}.\mathbf{a} & 1 \end{pmatrix}$, where a dot denotes d/du; this matrix is invertible so $\tilde{\boldsymbol{\gamma}}$ is a reparametrization of $\boldsymbol{\gamma}$.

5.3.4 $\boldsymbol{\sigma}_u = \dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}, \boldsymbol{\sigma}_v = \boldsymbol{\delta}$ (a dot denotes d/du) so $\dot{\boldsymbol{\delta}}(u)$ is perpendicular to the surface at $\boldsymbol{\sigma}(u, v) \iff \dot{\boldsymbol{\delta}}.(\dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}) = 0, \dot{\boldsymbol{\delta}}.\boldsymbol{\delta} = 0$. The second equation follows from $\parallel \boldsymbol{\delta} \parallel = 1$ so the two conditions are satisfied $\iff v = -(\dot{\boldsymbol{\gamma}}.\dot{\boldsymbol{\delta}})/ \parallel \dot{\boldsymbol{\delta}} \parallel^2$. Hence, $\boldsymbol{\Gamma}(u) = \boldsymbol{\gamma} - (\dot{\boldsymbol{\gamma}}.\dot{\boldsymbol{\delta}})\boldsymbol{\delta}/ \parallel \dot{\boldsymbol{\delta}} \parallel^2$. Using $\dot{\boldsymbol{\delta}}.\boldsymbol{\delta} = 0$ again, $\boldsymbol{\Gamma}.\dot{\boldsymbol{\delta}} = \dot{\boldsymbol{\gamma}}.\dot{\boldsymbol{\delta}} - (\dot{\boldsymbol{\gamma}}.\dot{\boldsymbol{\delta}})\dot{\boldsymbol{\delta}}.\dot{\boldsymbol{\delta}}/ \parallel \dot{\boldsymbol{\delta}} \parallel^2 = 0$.

5.3.5 The vector $\boldsymbol{\sigma}_u$ is tangent to the meridians, so a unit-speed curve $\boldsymbol{\gamma}$ is a loxodrome if $\dot{\boldsymbol{\gamma}}.\boldsymbol{\sigma}_u / \parallel \boldsymbol{\sigma}_u \parallel = \cos\alpha$, which gives $\dot{u} = \cosh u \cos\alpha$; since $\boldsymbol{\gamma}$ is unit-speed, $\dot{\boldsymbol{\gamma}} = (-\dot{u}\,\mathrm{sech}\,u \tanh u \cos v - \dot{v}\,\mathrm{sech}\,u \sin v, -\dot{u}\,\mathrm{sech}\,u \tanh u \sin v + \dot{v}\,\mathrm{sech}\,u \cos v, \dot{u}\,\mathrm{sech}^2 u)$ is a unit vector; this gives $\dot{u}^2 + \dot{v}^2 = \cosh^2 u$, so $\dot{v} = \pm\cosh u \sin\alpha$. The correponding curve in the uv-plane is given by $dv/du = \dot{v}/\dot{u} = \pm\tan\alpha$, and so is a straight line $v = \pm u \tan\alpha + c$, where c is a constant.

5.3.6 The ruling that makes an angle θ with the positive x-axis is given by $y = x\tan\theta$, $z = f(\theta)$, so a point at a distance u from the z-axis has coordinates $x = u\cos\theta, y = u\sin\theta, z = f(\theta)$. $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_\theta = (\frac{df}{d\theta}\sin\theta, -\frac{df}{d\theta}\cos\theta, u)$ and this is non-zero if $u \neq 0$ (i.e. if we exclude the line $\mathcal{L}$ itself from the surface).

5.3.7 (i) The assumption means that $\mathbf{N}$ is constant. Hence, $\boldsymbol{\sigma}.\mathbf{N}$ is constant, since $(\boldsymbol{\sigma}.\mathbf{N})_u = \boldsymbol{\sigma}_u.\mathbf{N} = 0$, etc. So $\boldsymbol{\sigma}$ is contained in a plane $\mathbf{v}.\mathbf{N} =$ constant.
(ii) If all the normal lines pass through a point $\mathbf{p}$, $\boldsymbol{\sigma} = \mathbf{p} + \lambda\mathbf{N}$ for some scalar λ. Then, $\boldsymbol{\sigma}_u = \lambda_u\mathbf{N} + \lambda\mathbf{N}_u$ so $0 = \boldsymbol{\sigma}_u.\mathbf{N} = \lambda_u$ since $\mathbf{N}_u.\mathbf{N} = 0$ as $\mathbf{N}$ is a unit vector. Similarly $\lambda_v = 0$. So λ is a constant and $\boldsymbol{\sigma}$ is part of the sphere $\parallel \mathbf{v} - \mathbf{p} \parallel = |\lambda|$.
(iii) By applying an isometry we can assume that the given straight line is the z-axis. Reparametrize the surface using polar coordinates in the xy-plane: $\boldsymbol{\sigma}(r,\theta) = (r\cos\theta, r\sin\theta, f(r,\theta))$, say. For some scalar μ, $\boldsymbol{\sigma} + \mu\mathbf{N}$ is the position vector of a point on the z-axis. If $\mathbf{N} = (a,b,c)$, setting the x- and y-components of $\boldsymbol{\sigma} + \mu\mathbf{N}$ equal to zero gives $\mu a = -r\cos\theta$, $\mu b = -r\sin\theta$, and hence either $\mu = 0$ (i.e., $r = 0$) or $a\sin\theta = b\cos\theta$. But $\boldsymbol{\sigma}_r \times \boldsymbol{\sigma}_\theta = (f_\theta \sin\theta - r f_r \cos\theta, -f_\theta\cos\theta - r f_r \sin\theta, r)$, so if $\mu \neq 0$ then

$$\sin\theta(r f_r \cos\theta - f_\theta \sin\theta) = \cos\theta(r f_r \sin\theta + f_\theta\cos\theta),$$

i.e. $f_\theta = 0$. We have now proved that $f_\theta = 0$ except possibly when $r = 0$; but then $f_\theta = 0$ for all (r,θ) as f is smooth (and hence continuous). Thus, f depends only on r, and hence $\boldsymbol{\sigma}$ is the surface of revolution obtained by rotating the curve $r \mapsto (r, 0, f(r))$ in the xz-plane around the z-axis.

5.3.8 Use the parametrization $\boldsymbol{\gamma}(u) = (\cos u, \sin u, 0)$ of the 'waist' of the hyperboloid. From Exercise 4.1.3, there is a straight line on the hyperboloid passing through $\boldsymbol{\gamma}(u)$, and it takes the form

$$(x - z)\cos\theta = (1 - y)\sin\theta, \ (x + z)\sin\theta = (1 + y)\cos\theta.$$

Substituting $x = \cos u$, $y = \sin u$, $z = 0$ gives $\theta = \frac{u}{2} + \frac{\pi}{4}$. The vector $\boldsymbol{\delta}(u) = (\cos 2\theta, \sin 2\theta, 1) = (-\sin u, \cos u, 1)$ is parallel to this line, so a parametrization of the hyperboloid as a ruled surface is

$$\boldsymbol{\sigma}(u,v) = (\cos u, \sin u, 0) + v(-\sin u, \cos u, 1) = (\cos u - v\sin u, \sin u + v\cos u, v).$$

Since $\dot{\boldsymbol{\gamma}}.\dot{\boldsymbol{\delta}} = 0$, the line of striction is $\boldsymbol{\gamma}$ (see the solution to Exercise 5.3.4).

5.3.9 We use the notation in Theorem 5.2.2.
(a) Types (vii) through (xi).
(b) Types (vi) and (x).
(c) Types (ii), (v) and (vi) through (xi).
(d) Type (i) if p^2, q^2, r^2 are not distinct; types (ii), (iii), (iv), (vi) and (vii) if $p^2 = q^2$; and types (x) and (xi).

5.3.10 Take the ruled surface to be $\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\delta}(u)$, where $\boldsymbol{\gamma}$ is unit-speed and $\boldsymbol{\delta}$ is a unit vector. Consider the ruling corresponding to $u = u_0$, say. Since $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (\dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}) \times \boldsymbol{\delta}$ (a dot denoting d/du, evaluated at $u = u_0$), the union of the normal lines along the ruling $u = u_0$ is the surface

$$\boldsymbol{\Sigma}(v, w) = \boldsymbol{\sigma}(u_0, v) + w(\dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}) \times \boldsymbol{\delta} = \mathbf{a} + v\mathbf{b} + w\mathbf{c} + vw\mathbf{d},$$

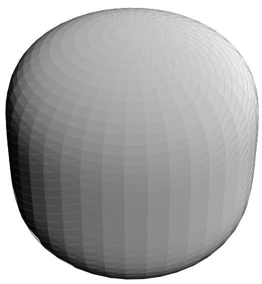
where $\mathbf{a} = \boldsymbol{\gamma}(u_0)$, $\mathbf{b} = \boldsymbol{\delta}(u_0)$, $\mathbf{c} = \dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta}$ and $\mathbf{d} = \dot{\boldsymbol{\delta}} \times \boldsymbol{\delta}$ are constant vectors.

By applying a translation we can assume that $\mathbf{a} = 0$. Then, by applying a rotation we can assume that $\mathbf{b} = (1, 0, 0)$ and that $\dot{\boldsymbol{\delta}}$ is parallel to $(0, 1, 0)$ (note that $\dot{\boldsymbol{\delta}}$ is perpendicular to $\boldsymbol{\delta}$). Then, $\mathbf{c}$ is parallel to the yz-plane and $\mathbf{d}$ is parallel to the z-axis.

If $\mathbf{d} = \mathbf{0}$, i.e. if $\dot{\boldsymbol{\delta}}(u_0) = 0$, $\boldsymbol{\Sigma}$ is a plane. If $\mathbf{d} = (0, 0, \lambda) \neq \mathbf{0}$ and $\mathbf{c} = (0, \mu, \nu)$, then $\boldsymbol{\Sigma}(v, w) = (v, \mu w, \nu w + \lambda vw)$. If $\mu = 0$ we have the xz-plane. Otherwise, the translation $x \mapsto x - \frac{\nu}{\lambda}$ followed by a rotation by $\pi/2$ around the z-axis takes $\boldsymbol{\Sigma}$ to a parametrization of the hyperbolic paraboloid $z = \frac{\lambda}{\mu}(x^2 - y^2)$.

5.4.1 Both surfaces are closed subsets of $\mathbb{R}^3$, as they are of the form $f(x, y, z) = 0$, where $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function (equal to $x^2 - y^2 + z^4 - 1$ and $x^2 + y^2 + z^4 - 1$ in the two cases). The surface in (i) is not bounded, and hence not compact, since it contains the point $(1, a^2, a)$ for all real numbers a; that in (ii) is bounded, and hence compact, since $x^2 + y^2 + z^4 = 1 \Longrightarrow -1 \leq x, y, z \leq 1$.

The surface in (ii) is obtained by rotating the curve $x^2 + z^4 = 1$ in the xz-plane around the z-axis:



5.4.2 A closed curve $\boldsymbol{\gamma}$ with period T can be identified with the unit circle by $\boldsymbol{\gamma}(t) \mapsto$

$(\cos(2\pi t/T), \sin(2\pi t/T))$. This gives rise to a diffeomorphism from the tube around $\boldsymbol{\gamma}$ to a tube around the circle, i.e. a torus. We have to make the tube have a sufficiently small radius as otherwise it might intersect itself and then it would not be a surface.

5.5.1 Both parts are geometrically obvious.

5.5.2 Let $(a, b, c) \in \mathbb{R}^3$ with a and b non-zero. Then $F_t(a, b, c) \to \infty$ as $t \to \infty$ and as t approaches p^2 and q^2 from the left; and $F_t(a, b, c) \to -\infty$ as $t \to -\infty$ and as t approaches p^2 and q^2 from the right. From this and the fact that $F_t(a, b, c) = 0$ is equivalent to a cubic equation for t, it follows that there exist unique numbers u, v, w with $u < p^2$, $p^2 < v < q^2$ and $q^2 < w$ such that $F_t(a, b, c) = 0$ when $t = u, v$ or w. The surfaces $F_u(x, y, z) = 0$ and $F_w(x, y, z) = 0$ are elliptic paraboloids and $F_v(x, y, z) = 0$ is a hyperbolic paraboloid, and we have shown that there is one surface of each type passing through each point (a, b, c).

To show that the system is orthogonal, note that $\boldsymbol{\nabla} F_t = \left(\frac{2x}{p^2-t}, \frac{2y}{q^2-t}, -2\right)$ so if (a, b, c) is a point at the intersection of the surfaces $F_u(x, y, z) = 1$ and $F_v(x, y, z) =$, we have

$$\begin{aligned}
\boldsymbol{\nabla} F_u . \boldsymbol{\nabla} F_v &= \frac{4a^2}{(p^2-u)(p^2-v)} + \frac{4b^2}{(q^2-u)(q^2-v)} + 4 \\
&= \frac{4(F_u(a, b, c) - F_v(a, b, c))}{u - v} = \frac{4-4}{u-v} = 0.
\end{aligned}$$

Thus, the first two families intersect orthogonally. Similarly for the other two pairs of families.

To parametrize these surfaces, write $F_t(x, y, z) = 0$ as the cubic equation

$$x^2(q^2 - t) + y^2(p^2 - t) - 2z(p^2 - t)(q^2 - t) + t(p^2 - t)(q^2 - t) = 0,$$

and note that the left-hand side must be equal to $(t-u)(t-v)(t-w)$; putting $t = p^2, q^2$ and then equating coefficients of t^2 (say) gives $x = \pm\sqrt{\frac{(p^2-u)(p^2-v)(p^2-w)}{q^2-p^2}}$, $y = \pm\sqrt{\frac{(q^2-u)(q^2-v)(q^2-w)}{p^2-q^2}}$, $z = \frac{1}{2}(u + v + w - p^2 - q^2)$.

5.5.3 (i) Let $U = xy - uz^2$, $V = x^2 + y^2 + z^2 - v$, $W = x^2 + y^2 + z^2 - w(x^2 - y^2)$. Then, $\boldsymbol{\nabla} U = (y, x, -2uz)$, $\boldsymbol{\nabla} V = (2x, 2y, 2z)$, $\boldsymbol{\nabla} W = (2(1-w)x, 2(1+w)y, 2z)$. Hence, at a point of intersection of the surfaces $U = 0$, $V = 0$, $W = 0$, we have $\boldsymbol{\nabla} U . \boldsymbol{\nabla} V = 4xy - 4uz^2 = 0$ by the equation $U = 0$;

$$\boldsymbol{\nabla} V . \boldsymbol{\nabla} W = 4((1-w)x^2 + (1+w)y^2 + z^2) = 4(x^2 + y^2 + z^2 - w(x^2 + y^2)) = 0$$

by the equation $V = 0$; and

$$\boldsymbol{\nabla} W . \boldsymbol{\nabla} U = 2(1-w)xy + 2(1+w)xy - 4uz^2 = 4(xy - uz^2) = 0$$

by the equation $U = 0$.
(ii) Let

$$U = yz - ux,\ V = \sqrt{x^2+y^2} + \sqrt{x^2+z^2} - v,\ W = \sqrt{x^2+y^2} - \sqrt{x^2+z^2} - w.$$

Then, $\mathbf{\nabla}U = (-u, z, y)$, $\mathbf{\nabla}V = \left(\frac{x}{\sqrt{x^2+y^2}} + \frac{x}{\sqrt{x^2+z^2}}, \frac{y}{\sqrt{x^2+y^2}}, \frac{z}{\sqrt{x^2+z^2}}\right)$, $\mathbf{\nabla}W = \left(\frac{x}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+z^2}}, \frac{y}{\sqrt{x^2+y^2}}, -\frac{z}{\sqrt{x^2+z^2}}\right)$. Hence, at a point of intersection of the surfaces $U = 0$, $V = 0$ and $W = 0$, $\mathbf{\nabla}U.\mathbf{\nabla}V = (yz - ux)\left(\frac{1}{\sqrt{x^2+y^2}} + \frac{1}{\sqrt{x^2+z^2}}\right) = 0$ by the equation $U = 0$; $\mathbf{\nabla}V.\mathbf{\nabla}W = \frac{x^2}{x^2+y^2} - \frac{x^2}{x^2+z^2} + \frac{y^2}{x^2+y^2} - \frac{z^2}{x^2+z^2} = 0$; $\mathbf{\nabla}W.\mathbf{\nabla}U = (yz - ux)\left(\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{x^2+z^2}}\right) = 0$ by the equation $U = 0$.

5.5.4 Two families of plane curves $U(x, y) = u$ and $V(x, y) = v$ form an orthogonal system if $\mathbf{\nabla}U.\mathbf{\nabla}V = 0$ at each point of intersection of a curve of the first family with a curve of the second family. Here, $\mathbf{\nabla}U = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\right)$, etc.
(i) $U(x, y) = x$, $V(x, y) = y$, i.e. the two families of straight lines parallel to the y- and x-axis, respectively. The orthogonality is obvious.
(ii) $U(x, y) = y \cos u - x \sin u$, $V(x, y) = x^2 + y^2 - v^2$, i.e. the family of straight lines passing through the origin and the family of circles with centre the origin. The orthogonality is obvious.

5.5.5 The function $F_t(a, b)$ is a continuous function of t (for fixed non-zero real numbers a, b) in each of the open intervals $(-\infty, p^2)$, (p^2, q^2) and (q^2, ∞). Also, $F_t(a, b) \to 0$ as $t \to \pm\infty$, and $F_t(a, b) \to \pm\infty$ as t approaches p^2 or q^2 from the left ($+$ sign) or right ($-$ sign). It follows that there is at least one value of t in each of the open intervals $(-\infty, p^2)$ and (p^2, q^2) such that $F_t(a, b) = 1$. On the other hand, $F_t(a, b) = 1$ is equivalent to a quadratic equation for t which has at most two real roots. It follows that there are unique numbers $u \in (-\infty, p^2)$ and $v \in (p^2, q^2)$ such that $F_u(a, b) = F_v(a, b) = 1$. The conics $F_u(x, y) = 1$ and $F_v(x, y) = 1$ are ellipses and hyperbolas, respectively, and we have shown that there is one of each passing through each point on the plane that does not lie on one of the coordinate axes.

To show that this is an orthogonal system of curves, we calculate $\mathbf{\nabla}F_u.\mathbf{\nabla}F_v$ at a point (a, b) that lies on the conics $F_u = 1$ and $F_v = 1$. Since $\mathbf{\nabla}F_u = \left(\frac{2x}{p^2-u}, \frac{2y}{q^2-u}\right)$, we get

$$\begin{aligned}\mathbf{\nabla}F_u.\mathbf{\nabla}F_v &= \frac{4a^2}{(p^2-u)(p^2-v)} + \frac{4b^2}{(q^2-u)(q^2-v)}\\ &= \frac{4(F_u(a,b) - F_v(a,b))}{u-v} = \frac{4-4}{u-v} = 0.\end{aligned}$$

For the last part, let $F_t(x,y) = \frac{x^2}{p^2-t} - 2y + t$. Arguing as in Exercise 5.5.2, for any point $(a,b) \in \mathbb{R}^2$ not on one of the coordinate axes, there are unique numbers $u \in (-\infty, p^2)$ and $v \in (p^2, \infty)$ such that $F_u(a,b) = F_v(a,b) = 1$. Both systems of curves $F_u(x,y) = 1$ and $F_v(x,y) = 1$ are parabolas. At (a,b) we find that

$$\boldsymbol{\nabla} F_u . \boldsymbol{\nabla} F_v = \frac{4a^2}{(p^2-u)(p^2-v)} + 4 = \frac{4(F_u(a,b) - F_v(a,b))}{u-v} = \frac{4-4}{u-v} = 0,$$

showing that the parabolas form an orthogonal system.

5.5.6 Let $U(x,y) = u$ and $V(x,y) = v$ be an orthogonal system of plane curves (see Exercise 5.5.4). We will show that the system of surfaces $U(x,y) = u$, $V(x,y) = v$, $z = w$ is orthogonal. The first two families of surfaces consist of generalized cylinders, the third consists of parallel planes. Suppose that (a,b,c) is a point of intersection of a surface from each family. We have to prove that the vectors $\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, 0\right)$, $\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, 0\right)$ and $(0,0,1)$ are mutually perpendicular at (a,b,c). But it is obvious that the third vector is perpendicular to the other two, and the first two are perpendicular because the curves $U(x,y) = u$ and $V(x,y) = v$ intersect orthogonally at (a,b).

5.6.1 Let $F : W \to V$ be the smooth bijective map with smooth inverse $F^{-1} : V \to W$ constructed in the proof of Proposition 4.2.6. Then, $(u(t), v(t)) = F^{-1}(\boldsymbol{\gamma}(t))$ is smooth.

5.6.2 Suppose, for example, that $f_y \neq 0$ at (x_0, y_0). Let $F(x,y) = (x, f(x,y))$; then F is smooth and its Jacobian matrix $\begin{pmatrix} 1 & f_x \\ 0 & f_y \end{pmatrix}$ is invertible at (x_0, y_0). By the inverse function theorem, F has a smooth inverse G defined on an open subset of $\mathbb{R}^2$ containing $F(x_0, y_0) = (x_0, 0)$, and G must be of the form $G(x,z) = (x, g(x,z))$ for some smooth function g. Then $\boldsymbol{\gamma}(t) = (t, g(t,0))$ is a parametrization of the level curve $f(x,y) = 0$ containing (x_0, y_0).

The matrix $\begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \end{pmatrix}$ has rank 2 everywhere; suppose that, at some point (x_0, y_0, z_0) on the level curve, the 2×2 submatrix $\begin{pmatrix} f_y & f_z \\ g_y & g_z \end{pmatrix}$ is invertible. Then, the function $F(x,y,z) = (x, f(x,y,z), g(x,y,z))$ is smooth and its Jacobian matrix $\begin{pmatrix} 1 & 0 & 0 \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{pmatrix}$ is invertible at (x_0, y_0, z_0). Let $G(x,u,v) = (x, \varphi(x,u,v), \psi(x,u,v))$ be the smooth inverse of F defined near $(x_0, 0, 0)$. Then $\boldsymbol{\gamma}(t) = (t, \varphi(t,0,0), \psi(t,0,0))$ is a parametrization of the level curve $f(x,y,z) = g(x,y,z) = 0$ containing (x_0, y_0, z_0).

5.6.3 Let $\boldsymbol{\sigma}(u,v) = (f(u,v), g(u,v), h(u,v))$. The condition $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \neq \mathbf{0}$ at (u_0, v_0) means that the matrix $\begin{pmatrix} f_u & f_v \\ g_u & g_v \\ h_u & h_v \end{pmatrix}$ has rank 2 at (u_0, v_0), so at least one 2×2 submatrix is invertible, say $\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$. If $F(u,v) = (f(u,v), g(u,v))$, then as in the proof of Proposition 4.2.6 there is an open subset V of $\mathbb{R}^2$ containing $F(u_0, v_0)$ and an open subset W of U containing (u_0, v_0) such that $F : W \to V$ is bijective, in particular injective. Then the restriction of $\boldsymbol{\sigma}$ to W is injective.

5.6.4 Let $\boldsymbol{\sigma}(u,v) = (f(u,v), g(u,v), h(u,v))$. The condition that $\mathbf{N}(u_0, v_0)$ is not parallel to the xy-plane means that the matrix $\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$ is invertible at (u_0, v_0). If $F(u,v) = (f(u,v), g(u,v))$, then as in the proof of Proposition 4.2.6 there is an open subset V of $\mathbb{R}^2$ containing $F(u_0, v_0)$ and an open subset W of U containing (u_0, v_0) such that $F : W \to V$ is bijective with smooth inverse. If $F^{-1}(u,v) = (\alpha(u,v), \beta(u,v))$, then near (x_0, y_0, z_0) the surface coincides with the graph $z = h(\alpha(x,y), \beta(x,y))$. If $\mathbf{N}(u_0, v_0)$ is parallel to the xy-plane, then at least one of the other two 2×2 submatrices of the Jacobian matrix of $\boldsymbol{\sigma}(u,v)$ is invertible, and then the surface coincides near (x_0, y_0, z_0) with a graph of the form $x = \varphi(y,z)$ or $y = \varphi(x,z)$.

5.6.5 Let $\boldsymbol{\gamma}(t) = (\gamma_1(t), \ldots, \gamma_n(t))$. As $\boldsymbol{\gamma}$ is regular, $\dot{\boldsymbol{\gamma}}(t_0) \neq \mathbf{0}$, so at least one component of $\dot{\boldsymbol{\gamma}}(t_0)$ is non-zero, say $\dot{\gamma}_1(t_0)$ (the proof is the same in the other cases). By the inverse function theorem for real-valued function of one variable, there is a smooth function α, say, defined on an open interval containing $\gamma_1(t_0)$, and some $\epsilon > 0$ such that $\alpha(\gamma_1(t)) = t$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$. This implies that the restriction of γ_1 to the interval $(t_0 - \epsilon, t_0 + \epsilon)$ is injective. Hence, the restriction of $\boldsymbol{\gamma}$ to the same interval is injective.

Chapter 6

6.1.1 (i) Quadric cone $x^2 + z^2 = y^2$; we have

$$\begin{aligned}
\boldsymbol{\sigma}_u &= (\cosh u \sinh v, \cosh u \cosh v, \cosh u), \\
\boldsymbol{\sigma}_v &= (\sinh u \cosh v, \sinh u \sinh v, 0), \\
\| \boldsymbol{\sigma}_u \|^2 &= 2\cosh^2 u \cosh^2 v, \\
\boldsymbol{\sigma}_u . \boldsymbol{\sigma}_v &= 2 \sinh u \cosh u \sinh v \cosh v, \\
\| \boldsymbol{\sigma}_v \|^2 &= \sinh^2 u \cosh 2v,
\end{aligned}$$

and the first fundamental form is

$$2\cosh^2 u \cosh^2 v \, du^2 + \sinh 2u \sinh 2v \, du dv + \sinh^2 u \cosh 2v \, dv^2.$$

(ii) Paraboloid of revolution; $(2+4u^2)\,du^2 + 8uv\,dudv + (2+4v^2)\,dv^2$.
(iii) Hyperbolic cylinder; $(\cosh^2 u + \sinh^2 u)\,du^2 + dv^2$.
(iv) Paraboloid of revolution; $(1+4u^2)\,du^2 + 8uv\,dudv + (1+4v^2)\,dv^2$.

6.1.2 Applying a translation to a surface patch $\boldsymbol{\sigma}$ does not change $\boldsymbol{\sigma}_u$ or $\boldsymbol{\sigma}_v$. If P is a 3×3 orthogonal matrix, $(P\boldsymbol{\sigma})_u = P(\boldsymbol{\sigma}_u), (P\boldsymbol{\sigma})_v = P(\boldsymbol{\sigma}_v)$, and P preserves dot products ($P(\mathbf{p}).P(\mathbf{q}) = \mathbf{p}.\mathbf{q}$ for all vectors $\mathbf{p},\mathbf{q}\in\mathbb{R}^3$). Applying the dilation $(x,y,z)\mapsto a(x,y,z)$, where a is a non-zero constant, multiplies $\boldsymbol{\sigma}$ by a and hence the first fundamental form by a^2.

6.1.3 Since both sides define bilinear forms on the tangent plane, it suffices to prove that the two sides agree when $\mathbf{v},\mathbf{w}$ belong to the basis $\{\boldsymbol{\sigma}_u,\boldsymbol{\sigma}_v\}$. This is easily checked using $du(\boldsymbol{\sigma}_u) = dv(\boldsymbol{\sigma}_v) = 1$, $du(\boldsymbol{\sigma}_v) = dv(\boldsymbol{\sigma}_u) = 0$.

6.1.4 By the chain rule, $\tilde{\boldsymbol{\sigma}}_{\tilde u} = \boldsymbol{\sigma}_u\frac{\partial u}{\partial \tilde u} + \boldsymbol{\sigma}_v\frac{\partial v}{\partial \tilde u}$, $\tilde{\boldsymbol{\sigma}}_{\tilde v} = \boldsymbol{\sigma}_u\frac{\partial u}{\partial \tilde v} + \boldsymbol{\sigma}_v\frac{\partial v}{\partial \tilde v}$, which gives $\tilde E = \tilde{\boldsymbol{\sigma}}_{\tilde u}.\tilde{\boldsymbol{\sigma}}_{\tilde u} = E\left(\frac{\partial u}{\partial \tilde u}\right)^2 + 2F\frac{\partial u}{\partial \tilde u}\frac{\partial v}{\partial \tilde u} + G\left(\frac{\partial v}{\partial \tilde u}\right)^2$. Similar expressions for $\tilde F$ and $\tilde G$ can be found; multiplying out the matrices shows that these formulas are equivalent to the matrix equation in the question.

Following the procedure given, $Edu^2 + 2Fdudv + Gdv^2 =$
$E\left(\frac{\partial u}{\partial \tilde u}d\tilde u + \frac{\partial u}{\partial \tilde v}d\tilde v\right)^2 + 2F\left(\frac{\partial u}{\partial \tilde u}d\tilde u + \frac{\partial u}{\partial \tilde v}d\tilde v\right)\left(\frac{\partial v}{\partial \tilde u}d\tilde u + \frac{\partial v}{\partial \tilde v}d\tilde v\right) + G\left(\frac{\partial v}{\partial \tilde u}d\tilde u + \frac{\partial v}{\partial \tilde v}d\tilde v\right)^2$. The coefficient of $d\tilde u^2$ is $E\left(\frac{\partial u}{\partial \tilde u}\right)^2 + 2F\frac{\partial u}{\partial \tilde u}\frac{\partial v}{\partial \tilde u} + G\left(\frac{\partial v}{\partial \tilde u}\right)^2$, which agrees with the expression for $\tilde E$ found above. Similarly for $\tilde F$ and $\tilde G$.

6.1.5 (i) $\Longleftrightarrow$ (ii): $E_v = G_u = 0 \Longleftrightarrow \boldsymbol{\sigma}_u.\boldsymbol{\sigma}_{uv} = \boldsymbol{\sigma}_v.\boldsymbol{\sigma}_{uv} = 0 \Longleftrightarrow \boldsymbol{\sigma}_{uv}$ is parallel to $\mathbf{N}$. Consider the quadrilateral bounded by the parameter curves $u = u_0, u = u_1, v = v_0, v = v_1$. The length of the side given by $u = u_0$ is $\int_{v_0}^{v_1} \parallel (\boldsymbol{\sigma}_v(u_0,v) \parallel dv = \int_{v_0}^{v_1}\sqrt{G(u_0,v)}dv$.
(i) $\Longrightarrow$ (iii): If $G_u = 0$, G depends only on v so the last integral is unchanged when u_0 is replaced by u_1. So the two sides $u = u_0$ and $u = u_1$ have the same length; and similarly for the other two sides.
(iii) $\Longrightarrow$ (i): If the lengths are equal then $\int_{v_0}^{v_1}\sqrt{G(u,v)}dv$ is independent of u; differentiating with respect to u gives $\int_{v_0}^{v_1}\frac{G_u}{2\sqrt{G}}dv = 0$ for all v_0, v_1, so $G_u = 0$; and similarly $E_v = 0$.

Assuming conditions (i) - (iii) are satisfied, set $\tilde u = \int\sqrt{E(u)}\,du$, $\tilde v = \int\sqrt{G(v)}\,dv$. Then, $(u,v)\mapsto(\tilde u,\tilde v)$ is a reparametrization map because its Jacobian matrix $\begin{pmatrix}\sqrt{E} & 0\\ 0 & \sqrt{G}\end{pmatrix}$ has non-zero determinant $\sqrt{EG}$. The first fundamental form of the reparametrization $\tilde{\boldsymbol{\sigma}}(\tilde u,\tilde v)$ of $\boldsymbol{\sigma}(u,v)$ is $d\tilde u^2 + \frac{2F}{\sqrt{EG}}d\tilde u d\tilde v + d\tilde v^2$. Now $\left|\frac{2F}{\sqrt{EG}}\right| < 1$ since $EG - F^2 > 0$, so there is a smooth function $\theta(\tilde u,\tilde v)$ with $0 < \theta < \pi$ such that $\cos\theta = \frac{2F}{\sqrt{EG}}$. This gives the first fundamental form as $d\tilde u^2 + 2\cos\theta d\tilde u d\tilde v + d\tilde v^2$.

Since $\tilde{u} = \frac{1}{2}(\hat{u}+\hat{v})$, $\tilde{v} = \frac{1}{2}(\hat{u}-\hat{v})$, the first fundamental form becomes

$$\frac{1}{4}(d\hat{u}+d\hat{v})^2 + \frac{1}{2}\cos\theta(d\hat{u}^2 - d\hat{v}^2) + \frac{1}{4}(d\hat{u}-d\hat{v})^2$$
$$= \frac{1}{2}(1+\cos\theta)d\hat{u}^2 + \frac{1}{2}(1-\cos\theta)d\hat{v}^2 = \cos^2\frac{\theta}{2}d\hat{u}^2 + \sin^2\frac{\theta}{2}d\hat{v}^2.$$

6.1.6 (i) $\boldsymbol{\sigma}_u = (\cos v, \sin v, 1/u)$, $\boldsymbol{\sigma}_v = (-u\sin v, u\cos v, 0)$ so $E = 1+u^{-2}$, $F = 0$, $G = u^2$, The surface is obtained by rotating the curve $z = \ln x$ in the xz-plane around the z-axis.
(ii) $\boldsymbol{\sigma}_u = (\cos v, \sin v, 0)$, $\boldsymbol{\sigma}_v = (-u\sin v, u\cos v, 1)$ so $E = 1$, $F = 0$, $G = 1+u^2$. This is a helicoid (Exercise 4.2.6).
(iii) $\boldsymbol{\sigma}_u = (\sinh u\cos v, \sinh u\sin v, 1)$, $\boldsymbol{\sigma}_v = (-\cosh u\sin v, \cosh u\cos v, 0)$ so $E = 1+\sinh^2 u = \cosh^2 u$, $F = 0$, $G = \cosh^2 u$. This is a catenoid (Exercise 5.3.1).

6.1.7 The first fundamental form is $2du^2 + u^2dv^2$. Hence, the length is

$$\int_0^\pi (2(\lambda e^{\lambda t})^2 + (e^{\lambda t})^2)^{1/2}dt = \frac{\sqrt{2\lambda^2+1}}{\lambda}(e^{\lambda\pi}-1).$$

A ruling can be parametrized by $u = t$, $v =$ constant, so the angle is

$$\cos^{-1}\left(\frac{2(\lambda e^{\lambda t} + u^2(0))}{\sqrt{(2\lambda^2+1)e^{2\lambda t}}\sqrt{2}}\right) = \cos^{-1}\sqrt{\frac{2\lambda^2}{2\lambda^2+1}},$$

which is constant and independent of the ruling.

6.1.8 Denoting d/du by a dot, $\boldsymbol{\sigma}_u = \dot{\boldsymbol{\gamma}} + v\dot{\mathbf{b}} = \mathbf{t} - \tau v\mathbf{n}$, $\boldsymbol{\sigma}_v = \mathbf{b}$, so since $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are perpendicular unit vectors, $E = 1+\tau^2v^2$, $F = 0$, $G = 1$.

6.1.9 From $E = \boldsymbol{\sigma}_u.\boldsymbol{\sigma}_u$ we get $E_u = \boldsymbol{\sigma}_{uu}.\boldsymbol{\sigma}_u + \boldsymbol{\sigma}_u.\boldsymbol{\sigma}_{uu} = 2\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_{uu}$. Hence, $\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_{uu} = \frac{1}{2}E_u$. Similarly, differentiating E with respect to v and G with respect to u and v gives $\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_{uv} = \frac{1}{2}G_u$, $\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_{uv} = \frac{1}{2}E_v$, $\boldsymbol{\sigma}_v.\boldsymbol{\sigma}_{vv} = \frac{1}{2}G_v$. Differentiating $F = \boldsymbol{\sigma}_u.\boldsymbol{\sigma}_v$ with respect to u gives $F_u = \boldsymbol{\sigma}_{uu}.\boldsymbol{\sigma}_v + \boldsymbol{\sigma}_u.\boldsymbol{\sigma}_{uv}$ and using the formula $\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_{uv} = \frac{1}{2}E_v$ gives $\boldsymbol{\sigma}_{uu}.\boldsymbol{\sigma}_v = F_u - \frac{1}{2}E_v$. The formula for $\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_{vv}$ is proved similarly.

6.2.1 The map is $\boldsymbol{\sigma}(u,v) \mapsto \left(u\sqrt{2}\cos\frac{v}{\sqrt{2}}, u\sqrt{2}\sin\frac{v}{\sqrt{2}}, 0\right) = \tilde{\boldsymbol{\sigma}}(u,v)$, say. The image of this map is the sector of the xy-plane whose polar coordinates (r,θ) satisfy $0 < \theta < \pi\sqrt{2}$. The first fundamental form of $\boldsymbol{\sigma}$ is $2\,du^2 + u^2\,dv^2$; $\tilde{\boldsymbol{\sigma}}_u = \left(\sqrt{2}\cos\frac{v}{\sqrt{2}}, \sqrt{2}\sin\frac{v}{\sqrt{2}}, 0\right)$, $\tilde{\boldsymbol{\sigma}}_v = \left(-u\sin\frac{v}{\sqrt{2}}, u\cos\frac{v}{\sqrt{2}}, 0\right)$, so $\parallel \tilde{\boldsymbol{\sigma}}_u \parallel^2 = 2$, $\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_v = 0$, $\parallel \boldsymbol{\sigma}_v \parallel^2 = u^2$ and the first fundamental form of $\tilde{\boldsymbol{\sigma}}$ is $2\,du^2 + u^2\,dv^2$.

6.2.2 No: the part of the ruling $(t,0,t)$ with $1 \le t \le 2$ (say) has length $\sqrt{2}$ and is mapped to the straight line segment $(t,0,0)$ with $1 \le t \le 2$, which has length 1.

6.2.3 A straightforward calculation shows that the first fundamental form of $\boldsymbol{\sigma}^t$ is $\cosh^2 u(du^2+dv^2)$; in particular, it is independent of t. Hence, $\boldsymbol{\sigma}(u,v)\mapsto\boldsymbol{\sigma}^t(u,v)$ is an isometry for all t. Taking $t=\pi/2$ gives the isometry from the catenoid to the helicoid; under this map, the parallels $u=$ constant on the catenoid go to circular helices on the helicoid, and the meridians $v=$ constant go to the rulings of the helicoid.

6.2.4 The line of striction is given by $v=-(\dot{\boldsymbol{\gamma}}.\dot{\boldsymbol{\delta}})\boldsymbol{\delta}/\parallel\dot{\boldsymbol{\delta}}\parallel^2$ (Exercise 5.3.4), where in this case $\boldsymbol{\delta}=\dot{\boldsymbol{\gamma}}$. Since $\boldsymbol{\gamma}$ is unit-speed, $\dot{\boldsymbol{\gamma}}.\ddot{\boldsymbol{\gamma}}=0$ so $v=0$ and we get the curve $\boldsymbol{\gamma}$ itself. For the second part, we can assume that $u_0=0$ and by applying an isometry of $\mathbb{R}^3$ that $\boldsymbol{\gamma}(0)=\mathbf{0},\mathbf{t}(0)=\mathbf{i},\mathbf{n}(0)=\mathbf{j},\mathbf{b}(0)=\mathbf{k}$ (in the usual notation). Then, using Frenet-Serret, $\ddot{\boldsymbol{\gamma}}(0)=\kappa(0)\mathbf{j}$, $\dddot{\boldsymbol{\gamma}}(0)=(-\kappa(0)^2,\dot{\kappa}(0),\kappa(0)\tau(0))$ so, neglecting higher powers of u in each component, $\boldsymbol{\gamma}(u)=\boldsymbol{\gamma}(0)+u\dot{\boldsymbol{\gamma}}(0)+\frac{1}{2}\ddot{\boldsymbol{\gamma}}(0)u^2+\frac{1}{6}\dddot{\boldsymbol{\gamma}}(0)u^3+\ldots=(u,\frac{1}{2}\kappa(0)u^2,\frac{1}{6}\kappa(0)\tau(0)u^3)$. The intersection of the surface with the plane perpendicular to $\mathbf{t}(0)=\mathbf{i}$ is given by setting the x-component of $\boldsymbol{\sigma}(u,v)$ equal to zero. This gives $v=-u+$ higher terms, so neglecting such terms, $u=-v$. Then the intersection is $\boldsymbol{\Gamma}(v)=\boldsymbol{\sigma}(-v,v)=\boldsymbol{\gamma}(-v)+v\dot{\boldsymbol{\gamma}}(-v)=(0,-\frac{1}{2}\kappa(0)v^2,\frac{1}{3}\kappa(0)\tau(0)v^3)$.

6.2.5 A generalized cylinder can be parametrized by $\boldsymbol{\sigma}(u,v)=\boldsymbol{\gamma}(u)+v\mathbf{a}$, where we can assume (see Exercise 5.3.3) that $\boldsymbol{\gamma}$ is unit-speed, $\mathbf{a}$ is a unit vector, and $\dot{\boldsymbol{\gamma}}.\mathbf{a}=0$, where the dot denotes d/du. Then, $\boldsymbol{\sigma}_u=\dot{\boldsymbol{\gamma}}$, $\boldsymbol{\sigma}_v=\mathbf{a}$ so the first fundamental form is du^2+dv^2. This is the same as the first fundamental form of the plane $(u,v,0)$, so by Corollary 6.2.3 the map $\boldsymbol{\sigma}(u,v)\mapsto(u,v,0)$ is a local isometry.

By applying an isometry of $\mathbb{R}^3$ (a translation), we can assume that the vertex of the cone is the origin. It can then be parametrized by $\boldsymbol{\sigma}(u,v)=v\boldsymbol{\gamma}(u)$, where $\boldsymbol{\gamma}$ is a unit-speed curve on S^2. Then, $\boldsymbol{\sigma}_u=v\dot{\boldsymbol{\gamma}}$, $\boldsymbol{\sigma}_v=\boldsymbol{\gamma}$ so the first fundamental form is $v^2du^2+dv^2$. This is the same as the first fundamental form of the plane in polar coordinates $(v\cos u, v\sin u, 0)$.

6.2.6 Assume that f satisfies the given condition and denote d/du by a dot. Define $g(u)=\int\sqrt{1-\dot{f}^2}\,du$; then g is smooth since the term inside the square root is always >0. The curve $\boldsymbol{\gamma}(u)=(f(u),0,g(u))$ in the xz-plane is unit-speed, and by Example 6.1.3 if we rotate $\boldsymbol{\gamma}$ around the z-axis, the resulting surface of revolution has first fundamental form $du^2+f(u)^2dv^2$. By Corollary 6.2.3, the given surface is locally isometric to this surface of revolution.

6.2.7 Define $\tilde{u}=\int\sqrt{U(u)}\,du$. The given condition on U implies that $U(u)>0$ for all values of u, so $\tilde{u}$ is a smooth function of u and $d\tilde{u}/du$ is never zero. It follows that $(u,v)\mapsto(\tilde{u},v)$ is a reparametrization map. The first fundamental form in terms of $(\tilde{u},v)$ is $d\tilde{u}^2+\tilde{U}(\tilde{u})dv^2$, where $\tilde{U}(\tilde{u})=U(u)$. By Exercise 6.2.6, this surface patch is locally isometric to a surface of revolution if $\left|\frac{df}{d\tilde{u}}\right|<1$ for all

values of $\tilde{u}$, where $f = \sqrt{\tilde{U}}$. But,

$$\frac{df}{d\tilde{u}} = \frac{1}{2\sqrt{\tilde{U}}}\frac{d\tilde{U}}{d\tilde{u}} = \frac{1}{2\sqrt{U}}\frac{dU/du}{d\tilde{u}/du} = \frac{1}{2U}\frac{dU}{du}.$$

6.3.1 If the first fundamental forms of two surfaces are equal, they are certainly proportional, so any isometry is a conformal map. Stereographic projection is a conformal map from S^2 to the plane, but it is not an isometry since $\lambda \neq 1$ (see Example 6.3.5).

6.3.2 The first fundamental form of the given surface patch is $(1+u^2+v^2)^2(du^2+dv^2)$; this is a multiple of $du^2 + dv^2$ so the patch is conformal.

6.3.3 The first fundamental form of $\tilde{\boldsymbol{\sigma}}(u,v)$ is $\left(\frac{d\psi}{du}\right)^2 du^2 + \cos^2\psi(u)dv^2$. So $\tilde{\boldsymbol{\sigma}}$ is conformal $\iff d\psi/du = \pm\cos\psi$. Taking the plus sign, we get $u = \int \sec\psi d\psi = \ln(\sec\psi + \tan\psi)$, so $\frac{1+\sin\psi}{\cos\psi} = e^u$. Then $2\cosh u = e^u + e^{-u} = \frac{1+\sin\psi}{\cos\psi} + \frac{\cos\psi}{1+\sin\psi} = 2\sec\psi$. Hence, $\cos\psi = \mathrm{sech} u, \sin\psi = \tanh u$ and $\tilde{\boldsymbol{\sigma}}(u,v)$ is the patch in Exercise 5.3.2.

6.3.4 Φ is conformal if and only if $f_u^2 + g_u^2 = f_v^2 + g_v^2$ and $f_u f_v + g_u g_v = 0$. Let $z = f_u + ig_u$, $w = f_v + ig_v$; then Φ is conformal if and only if $z\bar{z} = w\bar{w}$ and $z\bar{w} + \bar{z}w = 0$, where the bar denotes complex conjugate; if $z = 0$, then $w = 0$ and all four equations are certainly satisfied; if $z \neq 0$, the equations give $z^2 = -w^2$, so $z = \pm iw$; these are easily seen to be equivalent to the first pair of equations in the statement of the exercise if the sign is $+$, and to the second pair if the sign is $-$. We have $\det(J(\Phi)) = \begin{vmatrix} f_u & g_u \\ f_v & g_v \end{vmatrix} = \pm(f_u^2 + f_v^2)$, with a plus sign if the first pair of equations hold and a minus sign if the second pair of equations hold.

6.3.5 Let $\mathcal{S}$ be an orientable surface. Fix a smooth choice of unit normal at each point of $\mathcal{S}$, and let $\mathcal{A}$ be the atlas for $\mathcal{S}$ consisting of all the surface patches for $\mathcal{S}$ whose standard unit normal agrees with the chosen normal. On the other hand, by Theorem 6.3.6 $\mathcal{S}$ has an atlas consisting of conformal parametrizations; let $\tilde{\mathcal{A}}$ be the maximal such atlas (i.e. the set of all conformal parametrizations of $\mathcal{S}$). Then, $\mathcal{A} \cap \tilde{\mathcal{A}}$ is an atlas for $\mathcal{S}$. Indeed, if $\mathbf{p} \in \mathcal{S}$, let $\boldsymbol{\sigma}$ be any conformal parametrization of $\mathcal{S}$ containing $\mathbf{p}$. If $\boldsymbol{\sigma}$ has the wrong orientation (so that $\boldsymbol{\sigma} \notin \tilde{\mathcal{A}}$), then $\tilde{\boldsymbol{\sigma}}(u,v) = \boldsymbol{\sigma}(-u,v)$ is a conformal parametrization containing $\mathbf{p}$ that has the correct orientation. Thus, in any case there is a surface patch of $\mathcal{S}$ containing $\mathbf{p}$ that is both conformal and correctly oriented. Let Φ be the transition map between two of the patches in the atlas $\mathcal{A} \cap \tilde{\mathcal{A}}$. Then, Φ is a conformal diffeomorphism between open subsets of $\mathbb{R}^2$. By Exercise 6.3.4, Φ is either holomorphic or anti-holomorphic, and in the latter case $\det(J(\Phi)) < 0$, contradicting the fact that Φ is the transition map between two correctly oriented surface patches. Hence, Φ must be holomorphic.

6.3.6 Following Example 6.3.5, we find $\tilde{\Pi}(x, y, z) = (\frac{x}{z+1}, \frac{y}{z+1}, 0)$. Identifying $(u, v) \in \mathbb{R}^2$ with $w = u + iv \in \mathbb{C}$, we find $\tilde{\boldsymbol{\sigma}}_1(w) = (\frac{2w}{|w|^2+1}, \frac{1-|w|^2}{1+|w|^2})$. Then $\boldsymbol{\sigma}_1(w) = \tilde{\boldsymbol{\sigma}}_1(1/\bar{w})$, so the transition map is $w \mapsto 1/\bar{w}$. This is not holomorphic, so the atlas $\{\boldsymbol{\sigma}_1, \tilde{\boldsymbol{\sigma}}_1\}$ does not give S^2 the structure of a Riemann surface. If $\hat{\boldsymbol{\sigma}}_1(w) = \tilde{\boldsymbol{\sigma}}_1(\bar{w})$, the transition map between $\boldsymbol{\sigma}_1$ and $\hat{\boldsymbol{\sigma}}_1$ is $w \mapsto 1/w$. This is holomorphic (when $w \neq 0$, which holds on the overlap of the two patches), so the atlas $\{\boldsymbol{\sigma}_1, \hat{\boldsymbol{\sigma}}_1\}$ gives S^2 the structure of a Riemann surface.

6.3.7 Any circle on S^2 is the intersection of S^2 with a plane, and so (see Appendix 2) has equation of the form $aw + \bar{a}\bar{w} + bz = c$, where $a \in \mathbb{C}$, $b, c \in \mathbb{R}$ are constants (and a and b are not both zero). Substituting $w = \frac{2\xi}{|\xi|^2+1}, z = \frac{|\xi|^2-1}{|\xi|^2+1}$ gives $(b-c)|\xi|^2 + 2a\xi + 2\bar{a}\bar{\xi} = b + c$, which is the equation of a Circle in $\mathbb{C}_\infty$ (a line if $b = c$, a circle otherwise). The converse is proved similarly.

6.3.8 The expression of the map $\Pi^{-1} \circ M \circ \Pi$ in terms of the atlas $\{\boldsymbol{\sigma}_1, \tilde{\boldsymbol{\sigma}}_1\}$ of S^2 in Exercise 6.3.6, which consists of conformal patches, is of the form $w \mapsto M(w), w \mapsto M(1/\bar{w}), w \mapsto \overline{M(w)}^{-1}$, or $w \mapsto \overline{M(1/\bar{w})}^{-1}$, i.e. a Möbius or conjugate-Möbius transformation. Since such transformations are conformal (Appendix 2), the result follows.

6.3.9 The first fundamental form is $(1 + \dot{f}^2)du^2 + u^2\, dv^2$, where a dot denotes d/du. So $\boldsymbol{\sigma}$ is conformal if and only if $\dot{f} = \pm\sqrt{u^2 - 1}$, i.e. if and only if $f(u) = \pm(\frac{1}{2}u\sqrt{u^2-1} - \frac{1}{2}\cosh^{-1} u) + c$, where c is a constant.

6.3.10 The first fundamental form is $(1 + 2v\dot{\boldsymbol{\gamma}}.\dot{\boldsymbol{\delta}} + v^2\dot{\boldsymbol{\delta}}.\dot{\boldsymbol{\delta}})du^2 + 2\dot{\boldsymbol{\gamma}}.\boldsymbol{\delta}\, du dv + dv^2$. So $\boldsymbol{\sigma}$ is conformal if and only if $1 + 2v\dot{\boldsymbol{\gamma}}.\dot{\boldsymbol{\delta}} + v^2\dot{\boldsymbol{\delta}}.\dot{\boldsymbol{\delta}} = 1$ and $\dot{\boldsymbol{\gamma}}.\boldsymbol{\delta} = 0$ for all u, v; the first condition gives $\dot{\boldsymbol{\delta}} = \mathbf{0}$, so $\boldsymbol{\delta}$ is constant, and the second condition then says that $\boldsymbol{\gamma}.\boldsymbol{\delta}$ is constant, say equal to d. Thus, $\boldsymbol{\sigma}$ is conformal if and only if $\boldsymbol{\delta}$ is constant and $\boldsymbol{\gamma}$ is contained in a plane $\mathbf{v}.\boldsymbol{\delta} = d$. In this case, $\boldsymbol{\sigma}$ is a generalized cylinder.

6.3.11 From $\boldsymbol{\sigma}_u^* = \frac{1}{\boldsymbol{\sigma}.\boldsymbol{\sigma}}\boldsymbol{\sigma}_u - \frac{2(\boldsymbol{\sigma}.\boldsymbol{\sigma}_u)}{(\boldsymbol{\sigma}.\boldsymbol{\sigma})^2}\boldsymbol{\sigma}$, we get (in the obvious notation)

$$E^* = \boldsymbol{\sigma}_u^*.\boldsymbol{\sigma}_u^* = \frac{E}{(\boldsymbol{\sigma}.\boldsymbol{\sigma})^2} - \frac{4(\boldsymbol{\sigma}.\boldsymbol{\sigma}_u)^2}{(\boldsymbol{\sigma}.\boldsymbol{\sigma})^3} + \frac{4(\boldsymbol{\sigma}.\boldsymbol{\sigma}_u)^2}{(\boldsymbol{\sigma}.\boldsymbol{\sigma})^3} = \frac{E}{\parallel \boldsymbol{\sigma} \parallel^4}.$$

Similarly, we find that $F^* = F/ \parallel \boldsymbol{\sigma} \parallel^4$, $G^* = G/ \parallel \boldsymbol{\sigma} \parallel^4$. Hence, the first fundamental form of $\boldsymbol{\sigma}^*$ is a multiple of that of $\boldsymbol{\sigma}$.

6.3.12 By Example 6.1.3, any surface of revolution has an atlas consisting of surface patches $\boldsymbol{\sigma}(u, v)$ whose first fundamental form is $du^2 + f(u)^2dv^2$ for some positive smooth function $f(u)$. Define $\tilde{u} = \int \frac{du}{f(u)}$. Then, the first fundamental form of the reparametrization $\tilde{\boldsymbol{\sigma}}(\tilde{u}, v)$ of $\boldsymbol{\sigma}(u, v)$ is $f^2(d\tilde{u}^2 + dv^2)$. This is a conformal surface patch.

6.4.1 Parametrize the paraboloid by $\boldsymbol{\sigma}(u, v) = (u, v, u^2 + v^2)$; its first fundamental form is $(1 + 4u^2)du^2 + 8uv\, dudv + (1 + 4v^2)dv^2$. Hence, the required area is

the integral $\iint \sqrt{1+4(u^2+v^2)}\,dudv$, taken over the disc $u^2+v^2<1$. Let $u=r\sin\theta, v=r\cos\theta$; then the area is $2\pi\int_0^1\sqrt{1+4r^2}r\,dr=\frac{\pi}{6}(5^{3/2}-1)$. This is less than the area 2π of the hemisphere.

6.4.2 If $\mathcal{S}$ is a sphere with centre the origin and radius R, the map $S^2\to\mathcal{S}$ given by $\mathbf{p}\mapsto R\mathbf{p}$ multiplies the first fundamental form by R^2, and so is conformal but multiplies areas by R^2. It follows from Theorem 6.4.7 that the sum of the angles of a spherical triangle of area A on $\mathcal{S}$ is $\pi+A/R^2$. In this case, R is the radius of the earth and A is $\geq$ the area of Australia, so the sum of the angles is $\geq\pi+(7\,500\,000)/(6\,500)^2=\pi+\frac{30}{169}$ radians. Hence, at least one angle of the triangle must be at least one third of this, i.e. $\pi+\frac{10}{169}$ radians.

6.4.3 Take a point $\mathbf{p}$ inside the polygon and join it to each vertex of the polygon by an arc of a great circle. This gives n triangles whose sides are arcs of great circles. The sum of their angles is the sum of their areas (i.e. the area of the polygon) minus $n\pi$ by Theorem 6.4.7, and is also the sum of the angles of the polygon plus 2π (the angle around $\mathbf{p}$).

6.4.4 The sum of the angles around any vertex is 2π, so the sum of the angles of all the polygons is $2\pi V$. By the preceding exercise, the sum of the angles of a polygon with n sides is $(n-2)\pi$ plus its area. Summing over all polygons gives $2\pi V=4\pi+\sum_{\text{polygons}}(n-2)\pi$, since the sum of the areas of all the polygons is the area 4π of the sphere. Since two polygons meet along each edge, $\sum_{\text{polygons}}n=2E$, and since there are F polygons altogether we get $2\pi V=2\pi E-2\pi F+4\pi$, which is equivalent to $V-E+F=2$.

6.4.5 (i) is obvious as a local isometry preserves E,F,G and hence $\sqrt{EG-F^2}$.

If $\tilde{E}=\lambda E$, $\tilde{F}=\lambda F$ and $\tilde{G}=\lambda G$, and if $\tilde{E}\tilde{G}-\tilde{F}^2=EG-F^2$, then $\lambda^2=1$ and so $\lambda=1$ (as $E,\tilde{E}$ are >0). This proves (ii).

The map from S^2 to the unit cylinder in the proof of Theorem 6.4.6 is an equiareal map that is not a local isometry.

6.4.6 Let $\boldsymbol{\sigma}:U\to\mathbb{R}^3$; f is equiareal $\iff$

$$\int\!\!\int_R(E_1G_1-F_1^2)^{1/2}\,dudv=\int\!\!\int_R(E_2G_2-F_2^2)^{1/2}\,dudv\quad\text{for all regions } R\subseteq U.$$

This holds $\iff$ the two integrands are equal everywhere, i.e. $\iff E_1G_1-F_1^2=E_2G_2-F_2^2$.

6.4.7 Since $\mathbf{N}$ is perpendicular to the tangent plane, $\mathbf{N}\times\boldsymbol{\sigma}_u$ is parallel to the tangent plane, and so $=\alpha\boldsymbol{\sigma}_u+\beta\boldsymbol{\sigma}_v$ for some α,β. Now $(\mathbf{N}\times\boldsymbol{\sigma}_u).\boldsymbol{\sigma}_u=0$, $(\mathbf{N}\times\boldsymbol{\sigma}_u).\boldsymbol{\sigma}_v=(\boldsymbol{\sigma}_u\times\boldsymbol{\sigma}_v).\mathbf{N}=\parallel\boldsymbol{\sigma}_u\times\boldsymbol{\sigma}_v\parallel\mathbf{N}.\mathbf{N}=\sqrt{EG-F^2}$ by Proposition 6.4.2. This gives the two equations $\alpha E+\beta F=0$, $\alpha F+\beta G=\sqrt{EG-F^2}$, which imply $\alpha=-F/\sqrt{EG-F^2}$, $\beta=E/\sqrt{EG-F^2}$. The formula for $\mathbf{N}\times\boldsymbol{\sigma}_v$ is proved similarly.

6.4.8 By Exercise 6.1.6(ii), the first fundamental form is $du^2 + (1 + u^2)dv^2$. So the area is

$$\int_0^{2\pi} \int_0^1 \sqrt{1+u^2}dudv = \pi(\sqrt{2} + \ln(1+\sqrt{2})).$$

(The integral can be evaluated by putting $u = \tan\theta$.)

6.4.9 If we count the m edges that meet at each vertex, and then sum over all vertices, we will have counted each edge twice, once from each end. Hence, $mV = 2E$. Similarly, counting the n edges of each polygon, and then summing over all the polygons, gives twice the total number of edges as each edge is an edge of two polygons. This gives $nF = 2E$.

From $V - E + F = 2$ and $V = 2E/m$, $F = 2E/n$ we get $E = \left(\frac{1}{m} + \frac{1}{n} - \frac{1}{2}\right)^{-1}$, and hence values for V and F. The desired inequality follows from the fact that V, E and F are all > 0.

The inequality can be written $(m-2)(n-2) < 4$. Thus, $m - 2$ and $n - 2$ are two positive integers whose product is less than 4, giving the 5 possibilities 1×1, 1×2, 2×1, 3×1, 1×3.

6.4.10 We can assume that the sphere is the unit sphere S^2. If such curves exist they would give a triangulation of S^2 with 5 vertices and $5 \times 4/2 = 10$ edges, hence $2 + 10 - 5 = 7$ polygons. Since each edge is an edge of two polygons and each polygon has at least 3 edges, $3F \leq 2E$; but $3 \times 7 > 2 \times 10$. If curves satisfying the same conditions exist in the plane, applying the inverse of the stereographic projection map (Example 6.3.5) would give curves satisfying the conditions on S^2, which we have shown is impossible.

6.4.11 Such a collection of curves would give a triangulation of the sphere with $V = 6$, $E = 9$, and hence $F = 5$. The total number of edges of all the polygons in the triangulation is $2E = 18$. Since exactly 3 edges meet at each vertex, going around each polygon once counts each edge 3 times, so there should be $18/3 = 6$ polygons, not 5.

6.4.12 Parametrize the surface by $\boldsymbol{\sigma}(u,v) = (\rho(u)\cos v, \rho(u)\sin v, \sigma(u))$, where $\boldsymbol{\gamma}(u) = (\rho(u), 0, \sigma(u))$. By Example 6.1.3, the first fundamental form is $du^2 + \rho(u)^2\,dv^2$, so the area is $\int\int \rho(u)\,dudv = 2\pi \int \rho(u)\,du$.

(i) Take $\rho(u) = \cos u$, $\sigma(u) = \sin u$, with $-\pi/2 \leq u \leq \pi/2$; so $2\pi \int_{-\pi/2}^{\pi/2} \cos u\,du = 4\pi$ is the area.

(ii) For the torus, the profile curve is $\boldsymbol{\gamma}(\theta) = (a + b\cos\theta, 0, b\sin\theta)$, but this is not unit-speed; a unit-speed reparametrisation is $\tilde{\boldsymbol{\gamma}}(u) = \left(a + b\cos\frac{u}{b}, 0, b\sin\frac{u}{b}\right)$ with $0 \leq u \leq 2\pi b$. So $2\pi \int_0^{2\pi b} \left(a + b\cos\frac{u}{b}\right)\,du = 4\pi^2 ab$ is the area.

6.4.13 Using the parametrization in Exercise 4.2.7, the first fundamental form is found

to be $((1-\kappa a\cos\theta)^2+\tau^2a^2)\,ds^2+2\tau a^2\,dsd\theta+a^2\,d\theta^2$, so the area is

$$\int_{s_0}^{s_1}\int_0^{2\pi} a(1-\kappa a\cos\theta)dsd\theta = 2\pi a(s_1-s_0).$$

6.4.14 If $E_1=\lambda E_2, F_1=\lambda F_2, G_1=\lambda G_2$, and if $E_1G_1-F_1^2=E_2G_2-F_2^2$, then $\lambda^2=1$ so $\lambda=1$ (since $\lambda>0$).

6.4.15 (i) Any linear map $\mathbb{R}^2\to\mathbb{R}^2$ is obviously smooth so M is a diffeomorphism if and only if M is bijective. This holds if and only if M takes a basis of $\mathbb{R}^2$, such as $\{\mathbf{i},\mathbf{j}\}$, to another basis. Hence, M is a diffeomorphism if and only if $\{\mathbf{u},\mathbf{v}\}$ is a basis of $\mathbb{R}^2$, i.e. if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.
(ii) M takes $(u,v)\in\mathbb{R}^2$ to $\boldsymbol{\sigma}(u,v)=u\mathbf{u}+v\mathbf{v}$. The first fundamental form of $\boldsymbol{\sigma}$ is $\parallel\mathbf{u}\parallel^2\,du^2+2(\mathbf{u}.\mathbf{v})dudv+\parallel\mathbf{v}\parallel^2\,dv^2$. This is equal to du^2+dv^2 if and only if $\parallel\mathbf{u}\parallel=\parallel\mathbf{v}\parallel=1$ and $\mathbf{u}.\mathbf{v}=0$, i.e. if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular unit vectors.
(iii) $\parallel\mathbf{u}\parallel^2\,du^2+2(\mathbf{u}.\mathbf{v})dudv+\parallel\mathbf{v}\parallel^2\,dv^2$ is a multiple of du^2+dv^2 if and only if $\parallel\mathbf{u}\parallel=\parallel\mathbf{v}\parallel$ and $\mathbf{u}.\mathbf{v}=0$, i.e. if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular vectors of equal length.
(iv) Since $E=\parallel\mathbf{u}\parallel^2$, $F=\mathbf{u}.\mathbf{v}$, $G=\parallel\mathbf{v}\parallel^2$, M is equiareal if and only if $EG-F^2=1$, i.e. $\parallel\mathbf{u}\parallel^2\parallel\mathbf{v}\parallel^2-(\mathbf{u}.\mathbf{v})^2=1$. From the proof of Proposition 6.4.2, the left-hand side is equal to $\parallel\mathbf{u}\times\mathbf{v}\parallel^2$. So M is equiareal if and only if $\parallel\mathbf{u}\times\mathbf{v}\parallel=1$.

6.4.16 From the solution to Exercise 6.3.9, the first fundamental form of $\boldsymbol{\sigma}$ is (denoting d/du by a dot) $(1+\dot f^2)du^2+u^2\,dv^2$. So $\boldsymbol{\sigma}$ is equiareal if and only if $u^2(1+\dot f^2)=1$, i.e. $\frac{df}{du}=\pm\frac{\sqrt{1-u^2}}{u}$. Thus,

$$f(u)=\pm\int\frac{\sqrt{1-u^2}}{u}du=\pm\left(\sqrt{1-u^2}-\cosh^{-1}\left(\frac{1}{u}\right)\right)$$

(up to adding an arbitrary constant). This is a parametrization of the *tractrix*, so $\boldsymbol{\sigma}$ is equiareal $\Longleftrightarrow$ $\boldsymbol{\sigma}$ is a reparametrization of the *pseudosphere* (see §8.3).

6.4.17 The first fundamental form of $\boldsymbol{\sigma}_3$ is $\cos^2\theta d\theta^2+2\dot f\cos\theta d\theta d\varphi+(1+\dot f^2)d\varphi^2$, where the dot denotes $d/d\theta$. With $E=\cos^2\theta$, $F=\dot f\cos\theta$, $G=1+\dot f^2$, $EG-F^2=\cos^2\theta$. As this does not depend on f, the fact that the map $\boldsymbol{\sigma}_1(\theta,\varphi)\mapsto\boldsymbol{\sigma}_3(\theta,\varphi)$ is equiareal follows from the fact, proved in Theorem 6.4.6, that the map $\boldsymbol{\sigma}_1(\theta,\varphi)\mapsto\boldsymbol{\sigma}_2(\theta,\varphi)$ is equiareal.

6.5.1 If the internal angles are equal to α, Theorem 6.4.7 gives $3\alpha-\pi=4\pi/4$, so $\alpha=2\pi/3$. Corollary 6.5.6 then gives the length of a side as $A=\cos^{-1}(-1/3)$.

6.5.2 Using the notation following Proposition 6.5.8, there is a rotation R_1 of S^2 that takes $\mathbf{a}'$ to $\mathbf{a}$; then a further rotation R_2 around the diameter through $\mathbf{a}$ that

makes the side through $\mathbf{a}$ and $R_1(\mathbf{b}')$ coincide with the side through $\mathbf{a}$ and $\mathbf{b}$. By Corollary 6.5.6 the two triangles have sides of the same length, so we must have $\mathbf{b} = R_2R_1(\mathbf{b}')$. If $\mathbf{c}$ and $\mathbf{c}'$ are on the same side of the plane containing the side through $\mathbf{a}$ and $\mathbf{b}$, we shall then have $\mathbf{c} = R_2R_1(\mathbf{c}')$ and the isometry R_2R_1 takes the triangle with vertices $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ to the triangle with vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$; if they are on opposite sides the isometry $R_3R_2R_1$ does this, where R_3 is reflection in the plane containing the side through $\mathbf{a}$ and $\mathbf{b}$.

6.5.3 By applying an isometry of $\mathbb{R}^3$, which leaves lengths and areas unchanged, we can assume that $\mathbf{p}$ is the north pole $(0,0,1)$. The spherical circle of radius R and centre $\mathbf{p}$ is then the circle of latitude $\varphi = \pi/2 - R$ (Example 4.1.4), which is a circle of radius $\sin R$. The area inside it is, by Example 6.1.3 and Proposition 6.4.2, $\int_0^{2\pi}\int_{\pi/2-R}^{\pi/2} \cos\theta d\theta d\varphi = 2\pi(1-\cos R)$. The maximum value of R is π; if $\pi/2 \le R \le \pi$, one replaces R by $\pi - R$ in (i) and (ii).

6.5.4 (i) If $M'(w) = \frac{a'w+b'}{c'w+d'}$ is another unitary Möbius transformation, $(M' \circ M)(w) = \frac{Aw+B}{Cw+D}$, where $A = a'a + b'c, B = a'b + b'd, C = c'a + d'c, D = c'b + d'd$. Thus, $\bar{A} = \bar{a}'\bar{a} + \bar{b}'\bar{c} = d'd + (-c')(-b) = D$ and similarly $C = -\bar{B}$. Inverses are dealt with similarly.

(ii) Denoting $(x,y,z) \in \mathbb{R}^3$ by (ξ, z) with $\xi = x + iy \in \mathbb{C}$, the plane through the origin perpendicular to (a,b) is $\bar{w}a + w\bar{a} + 2bz = 0$, and reflection in it is $F(\xi,z) = (\xi,z) - 2\frac{\bar{w}a+w\bar{a}+2bz}{|a|^2+b^2}(a,b)$. Taking $\xi = \frac{2w}{|w|^2+1}$, $z = \frac{|w^2-1}{|w|^2+1}$, we find that $F(\xi,z) = (\xi',z')$, where $\xi' = \frac{2(|a|^2+b^2)w-2a(\bar{w}a+w\bar{a}+b(|w|^2-1))}{(|w|^2+1)(|a|^2+b^2)}$, $z' = \frac{(|a|^2+b^2)(|w|^2-1)-2b(\bar{w}a+w\bar{a}+b(|w|^2-1))}{(|w|^2+1)(|a|^2+b^2)}$, giving $w' = \frac{\xi'}{1-z'} = \frac{-ab|w|^2+b^2w-a^2\bar{w}+ab}{b^2|w|^2+b\bar{a}w+ba\bar{w}+|a|^2} = \frac{(-a\bar{w}+b)(bw+a)}{(b\bar{w}+\bar{a})(bw+a)} = \frac{-a\bar{w}+b}{b\bar{w}+\bar{a}}$.

(iii) By Proposition 6.5.7, if F is any isometry of S^2,

$$F_\infty = (M_1 \circ J) \circ (M_2 \circ J) \circ \cdots \circ (M_k \circ J) \quad \text{for some } k.$$

Since $J \circ M \circ J$ is easily seen to be a unitary Möbius transformation if M is one, part (i) implies that F_∞ is a unitary Möbius transformation if k is even, and of the form $M \circ J$ with M unitary Möbius if k is odd.

(iv) If $a \in \mathbb{C}, b \in \mathbb{R}$, call the unitary Möbius transformation $M(w) = \frac{aw+b}{-bw+\bar{a}}$ *special unitary*. Then $M = F_\infty \circ J$ where F is as in (ii). Since $J = R_\infty$ where R is reflection in the yz-plane, $M = (F \circ R)_\infty$ corresponds to the isometry $F \circ R$ of S^2. It therefore suffices to prove that every unitary Möbius transformation is a composite of finitely-many special unitary Möbius transformations. If $M'(w) = \frac{Aw+B}{-\bar{B}w+\bar{A}}$ is any unitary Möbius tranformation, where $A, B \in \mathbb{C}$, let $B = be^{i\theta}$ with $b, \theta \in \mathbb{R}$. Then $M' = \rho \circ M \circ \rho^{-1}$, where $\rho(w) = e^{i\theta}$ and $M(w) = \frac{Aw+b}{-bw+\bar{A}}$ are both special unitary Möbius transformations ($\rho(w) = \frac{aw+b}{-bw+\bar{a}}$ with $a = e^{i\theta/2}, b = 0$).

6.5.5 The dilation $\mathbf{v} \mapsto R^{-1}\mathbf{v}$, followed by a suitable translation, takes the sphere $\mathcal{S}$ of radius R to S^2. It takes a triangle on $\mathcal{S}$ with sides of length A, B, C to a triangle

on S^2 with sides of length $A/R, B/R, C/R$ and the same angles. Applying the cosine rule to the triangle on S^2 gives $\cos\frac{C}{R} = \cos\frac{A}{R}\cos\frac{B}{R} + \cos\gamma\sin\frac{A}{R}\sin\frac{B}{R}$, where γ is the angle opposite the side of length C (and two similar formulas).

If R is large, $\cos\frac{x}{R}$ is approximately equal to $1 - \frac{x^2}{2R^2}$ and $\sin\frac{x}{R}$ is approximately equal to $\frac{x}{R}$. Substituting into the cosine rule gives

$$1 - \frac{C^2}{2R^2} = \left(1 - \frac{A^2}{2R^2}\right)\left(1 - \frac{B^2}{2R^2}\right) + \frac{AB}{R^2}\cos\gamma,$$

which gives $C^2 = A^2 + B^2 - 2AB\cos\gamma - \frac{A^2B^2}{2R^2}$. Letting $R \to \infty$ gives the usual cosine rule for a plane triangle with sides of length A, B, C and with γ the angle opposite the side of length C. This happens because when $R \to \infty$ and A, B, C are fixed, the triangle becomes more and more nearly planar. (When standing at a point on a sphere of very large radius, one seems to be standing on a plane - this is why the Ancients thought the Earth is flat!)

6.5.6 Let $\mathbf{n}$, $\mathbf{a}$, $\mathbf{b}$ be the North Pole, Athens and Bombay, respectively, and let $R =$ 6500km be the radius of the Earth. The angle at $\mathbf{n}$ of the spherical triangle with vertices $\mathbf{n}, \mathbf{a}, \mathbf{b}$ is $73° - 24° = 49°$, and the lengths of the sides opposite $\mathbf{b}$ and $\mathbf{a}$ are obtained by multiplying by R the radian equivalents of $90° - 19° = 71°$ and $90° - 38° = 52°$, respectively. If d is the spherical distance between Athens and Bombay, applying the cosine rule in the preceding exercise thus gives

$$\cos\frac{d}{R} = \cos 52° \cos 71° + \sin 52° \sin 71° \cos 49° = 0.6892.$$

Hence, $d/R = 0.8104$ (radians) and $d = 0.8104R = 5267$km.

6.5.7 Drawing the diagonal of the square gives two spherical triangles with angles $\alpha/2, \alpha/2$ and α, the sides opposite these angles being A, A and D (the length of the diagonal). Applying the cosine rule to one of these triangles gives

$$\begin{aligned}\cos A &= \cos A\cos D + \sin A\sin D\cos\frac{1}{2}\alpha,\\ \cos D &= \cos^2 A + \sin^2 A\cos\alpha.\end{aligned}$$

Using the sine rule (Proposition 6.5.3(ii)) gives $\sin D = \sin A\sin\alpha/\sin\frac{1}{2}\alpha = 2\sin A\cos\frac{1}{2}\alpha$. Substituting the formulas for $\cos D$ and $\sin D$ into the first equation above gives

$$\cos A = \cos^3 A + \cos A\sin^2 A\cos\alpha + 2\sin^2 A\cos^2\frac{1}{2}\alpha.$$

Hence, $\cos A\sin^2 A(1 - \cos\alpha) = 2\sin^2 A\cos^2\frac{1}{2}\alpha$, and this gives the desired formula for $\cos A$.

6.5.8 (i) This is obvious by the sine rule.
(ii) This follows immediately from Corollary 6.5.6. Similarly one proves that $\cos\beta + \cos\alpha\cos\gamma = \lambda\sin\gamma\sin A\cos B$.
(iii) By (ii),

$$\begin{aligned}(\cos\alpha + \cos\beta)(1 + \cos\gamma) &= \cos\alpha + \cos\beta\cos\gamma + \cos\beta + \cos\alpha\cos\gamma \\ &= \lambda\sin\gamma\sin B\cos A + \lambda\sin\gamma\sin A\cos B\end{aligned}$$

by (ii), which $= \lambda\sin\gamma(\sin A\cos B + \cos A\sin B) = \lambda\sin\gamma\sin(A + B)$.
(iv) Writing the formula in (iii) in terms of half-angles gives

$$\cos\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta) = \lambda\tan\frac{1}{2}\gamma\sin\frac{1}{2}(A + B)\cos\frac{1}{2}(A + B).$$

Similarly, the first of the two equations in (i) can be written

$$\sin\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta) = \lambda\sin\frac{1}{2}(A + B)\cos\frac{1}{2}(A - B).$$

Dividing this equation by the preceding one gives the first formula in (iv). The second follows similarly from (iii) and the second equation in (i).
(v) Applying the first formula in (iv) to the polar triangle, we get

$$\tan\frac{1}{2}(\pi - A + \pi - B) = \frac{\cos\frac{1}{2}(\pi - \alpha - \pi + \beta)}{\cos\frac{1}{2}(\pi - \alpha + \pi - \beta)}\cot\frac{1}{2}(\pi - C),$$

i.e.,

$$\tan\frac{1}{2}(A + B) = \frac{\cos\frac{1}{2}(\alpha - \beta)}{\cos\frac{1}{2}(\alpha + \beta)}\tan\frac{1}{2}C.$$

Similarly,

$$\tan\frac{1}{2}(A - B) = \frac{\sin\frac{1}{2}(\alpha - \beta)}{\sin\frac{1}{2}(\alpha + \beta)}\tan\frac{1}{2}C.$$

6.5.9 If $\mathbf{a}' \neq \pm\mathbf{a}$ a rotation through an angle $\frac{1}{2}d_{S^2}(\mathbf{a}, \mathbf{a}')$ around the line through the origin parallel to $\mathbf{a} \times \mathbf{a}'$ takes $\mathbf{a}'$ to $\mathbf{a}$. If $\mathbf{a}' = -\mathbf{a}$ the same thing is achieved by applying the isometry $\mathbf{v} \mapsto -\mathbf{v}$. Thus, in all cases there is an isometry of S^2 that takes $\mathbf{a}'$ to $\mathbf{a}$. Since composites of isometries are isometries, it is sufficient to prove the result when $\mathbf{a}' = \mathbf{a}$.

If θ is the angle between the great circles passing through $\mathbf{a}$ and $\mathbf{b}$ and through $\mathbf{a}$ and $\mathbf{b}'$, a rotation by θ about the line through the origin and $\mathbf{a}$ fixes $\mathbf{a}$ and takes $\mathbf{b}'$ to a point $\mathbf{b}''$ on the great circle passing through $\mathbf{a}$ and $\mathbf{b}$. Since $\mathbf{b}$ and $\mathbf{b}'$ are equidistant from $\mathbf{a}$, so are $\mathbf{b}$ and $\mathbf{b}''$. It follows that either $\mathbf{b}'' = \mathbf{b}$ or $\mathbf{b}''$ is obtained

by reflecting $\mathbf{b}$ in the plane passing through the origin and $\mathbf{a}$ perpendicular to the great circle passing through $\mathbf{a}$ and $\mathbf{b}$. It is therefore sufficient to prove the theorem when $\mathbf{a} = \mathbf{a}'$ and $\mathbf{b} = \mathbf{b}'$.

The great circle passing through $\mathbf{a}$ and $\mathbf{c}$ makes the same angle with that passing through $\mathbf{a}$ and $\mathbf{b}$ as does the great circle passing through $\mathbf{a}$ and $\mathbf{c}'$. Hence, either these great circles coincide or one is obtained from the other by reflecting in the plane containing $\mathbf{a}$, $\mathbf{b}$ and the origin. It therefore suffices to prove the result in the case where the two great circles coincide. Since $\mathbf{c}$ and $\mathbf{c}'$ are the same distance from $\mathbf{a}$ on the same great circle, either they coincide or $\mathbf{c}'$ is obtained from $\mathbf{c}$ by reflecting in the plane passing through the origin and $\mathbf{a}$ perpendicular to the great circle passing through $\mathbf{a}$ and $\mathbf{c}$.

Chapter 7

7.1.1 $\boldsymbol{\sigma}_u = (1, 0, 2u)$, $\boldsymbol{\sigma}_v = (0, 1, 2v)$, so $\mathbf{N} = \lambda(-2u, -2v, 1)$, where $\lambda = \frac{1}{\sqrt{1+4u^2+4v^2}}$; $\boldsymbol{\sigma}_{uu} = (0, 0, 2)$, $\boldsymbol{\sigma}_{uv} = \mathbf{0}$, $\boldsymbol{\sigma}_{vv} = (0, 0, 2)$, so $L = 2\lambda$, $M = 0$, $N = 2\lambda$, and the second fundamental form is $2\lambda(du^2 + dv^2)$.

7.1.2 $\boldsymbol{\sigma}_u.\mathbf{N}_u = -\boldsymbol{\sigma}_{uu}.\mathbf{N}$ (since $\boldsymbol{\sigma}_u.\mathbf{N} = 0$), so $\mathbf{N}_u.\boldsymbol{\sigma}_u = 0$; similarly, $\mathbf{N}_u.\boldsymbol{\sigma}_v = \mathbf{N}_v.\boldsymbol{\sigma}_u = \mathbf{N}_v.\boldsymbol{\sigma}_v = 0$; hence, $\mathbf{N}_u$ and $\mathbf{N}_v$ are perpendicular to both $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$, and so are parallel to $\mathbf{N}$. On the other hand, $\mathbf{N}_u$ and $\mathbf{N}_v$ are perpendicular to $\mathbf{N}$ since $\mathbf{N}$ is a unit vector. Thus, $\mathbf{N}_u = \mathbf{N}_v = \mathbf{0}$, and hence $\mathbf{N}$ is constant. Then, $(\boldsymbol{\sigma}.\mathbf{N})_u = \boldsymbol{\sigma}_u.\mathbf{N} = 0$, and similarly $(\boldsymbol{\sigma}.\mathbf{N})_v = 0$, so $\boldsymbol{\sigma}.\mathbf{N}$ is constant, say equal to d, and then $\boldsymbol{\sigma}$ is an open subset of the plane $\mathbf{v}.\mathbf{N} = d$.

7.1.3 From §4.5, $\tilde{\mathbf{N}} = \pm\mathbf{N}$, the sign being that of $\det(J)$. From $\tilde{\boldsymbol{\sigma}}_{\tilde{u}} = \boldsymbol{\sigma}_u \frac{\partial u}{\partial \tilde{u}} + \boldsymbol{\sigma}_v \frac{\partial v}{\partial \tilde{u}}$, $\tilde{\boldsymbol{\sigma}}_{\tilde{v}} = \boldsymbol{\sigma}_u \frac{\partial u}{\partial \tilde{v}} + \boldsymbol{\sigma}_v \frac{\partial v}{\partial \tilde{v}}$, we get

$$\tilde{\boldsymbol{\sigma}}_{\tilde{u}\tilde{u}} = \boldsymbol{\sigma}_u \frac{\partial^2 u}{\partial \tilde{u}^2} + \boldsymbol{\sigma}_v \frac{\partial^2 v}{\partial \tilde{u}^2} + \boldsymbol{\sigma}_{uu} \left(\frac{\partial u}{\partial \tilde{u}}\right)^2 + 2\boldsymbol{\sigma}_{uv} \frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{u}} + \boldsymbol{\sigma}_{vv} \left(\frac{\partial v}{\partial \tilde{u}}\right)^2.$$

So $\tilde{L} = \pm\left(L\left(\frac{\partial u}{\partial \tilde{u}}\right)^2 + 2M \frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{u}} + N\left(\frac{\partial v}{\partial \tilde{u}}\right)^2\right)$, since $\boldsymbol{\sigma}_u.\mathbf{N} = \boldsymbol{\sigma}_v.\mathbf{N} = 0$. This, together with similar formulas for $\tilde{M}$ and $\tilde{N}$, are equivalent to the matrix equation in the question.

7.1.4 Let $\boldsymbol{\sigma}$ be a surface patch, P a 3×3 orthogonal matrix, $\mathbf{a} \in \mathbb{R}^3$ a constant vector, and $\tilde{\boldsymbol{\sigma}} = P\boldsymbol{\sigma} + \mathbf{a}$. Then, $\tilde{\boldsymbol{\sigma}}_u = P\boldsymbol{\sigma}_u$, $\tilde{\boldsymbol{\sigma}}_v = P\boldsymbol{\sigma}_v$, so $\tilde{\boldsymbol{\sigma}}_u \times \tilde{\boldsymbol{\sigma}}_v = \pm\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v$ (Proposition A.1.6), the sign being $+$ if the isometry $\mathbf{v} \mapsto P\mathbf{v} + \mathbf{a}$ is direct and $-$ if it is opposite. It follows that (in the obvious notation), $\tilde{L} = \pm L, \tilde{M} = \pm M, \tilde{N} = \pm N$. The dilation $\mathbf{v} \mapsto a\mathbf{v}$, where a is a non-zero constant, multiplies $\boldsymbol{\sigma}$ by a and hence multiplies each of L, M, N by a.

7.1.5 $\boldsymbol{\sigma}_u = (\cos v, \sin v, 0)$, $\boldsymbol{\sigma}_v = (-u \sin v, u \cos v, 1)$, so $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (\sin v, -\cos v, u)$ and $\mathbf{N} = \frac{1}{\sqrt{1+u^2}}(\sin v, -\cos v, u)$. Then, $\boldsymbol{\sigma}_{uu} = \mathbf{0}$, $\boldsymbol{\sigma}_{uv} = (-\sin v, \cos v, 0)$, $\boldsymbol{\sigma}_{vv} =$

$(-u\cos v, -u\sin v, 0)$. This gives $L = \boldsymbol{\sigma}_{uu}.\mathbf{N} = 0$, $M = \boldsymbol{\sigma}_{uv}.\mathbf{N} = -\frac{1}{\sqrt{1+u^2}}$, $N = \boldsymbol{\sigma}_{vv}.\mathbf{N} = 0$. So the second fundamental form is $\frac{-2dudv}{\sqrt{1+u^2}}$.

7.1.6 The tangent developable is parametrized by $\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\mathbf{t}(u)$, where $\mathbf{t} = d\boldsymbol{\gamma}/du$. Using the standard notation and the Frenet-Serret equations, we find $\boldsymbol{\sigma}_u = \mathbf{t} + \kappa v\mathbf{n}$, $\boldsymbol{\sigma}_v = \mathbf{t}$, $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = -\kappa v\mathbf{b}$, $\mathbf{N} = -\frac{v}{|v|}\mathbf{b}$,

$$\boldsymbol{\sigma}_{uu} = \kappa\mathbf{n} + \frac{d\kappa}{du}v\mathbf{n} + \kappa v(-\kappa\mathbf{t} + \tau\mathbf{b}),\ \boldsymbol{\sigma}_{uv} = \kappa\mathbf{n},\ \boldsymbol{\sigma}_{vv} = \mathbf{0}.$$

Hence, $L = -\kappa\tau\frac{v^2}{|v|}$, $M = N = 0$. So the second fundamental form is $-\kappa\tau|v|du^2$. This vanishes at all points of the surface if and only if $\tau = 0$ at all points of $\boldsymbol{\gamma}$, which holds if and only if $\boldsymbol{\gamma}$ is planar (Proposition 2.3.3). In that case, the tangent developable is part of a plane. Thus, for tangent developables, the second fundamental form vanishes everywhere if and only if the surface is part of a plane. This is a special case of Exercise 7.1.2.

7.2.1 The paraboloid is the level surface $f = 0$ where $f(x,y,z) = z - x^2 - y^2$ and $\mathbf{N} = \frac{(f_x,f_y,f_z)}{\|(f_x,f_y,f_z)\|}$ is the corresponding unit normal. So $\mathcal{G}(x,y,z) = \frac{(-2x,-2y,z)}{(4x^2+4y^2+1)^{1/2}}$.

7.2.2 This is obvious since changing the orientation changes the Gauss map $\mathcal{G}$ to $-\mathcal{G}$.

7.2.3 The hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ is obtained by rotating the hyperbola $x^2 - z^2 = 1$ in the xz-plane around the z-axis. For this hyperbola, $\frac{dz}{dx} = \frac{x}{z} \to \pm 1$ as $x \to \pm\infty$. The unit normal to the hyperbola therefore makes an angle θ with the x-axis, where the angle θ takes all values in the range $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Rotating around the z-axis, it follows that the image of the Gauss map of the hyperboloid is the region of S^2 for which the latitude θ satisfies $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. This is an annulus on S^2.

The hyperboloid of two sheets $x^2 - y^2 - z^2 = 1$ is obtained by rotating the same hyperbola as above around the x-axis. The image of $\mathcal{G}$ is the union of two 'caps' on S^2, namely the parts of the sphere $x^2 + y^2 + z^2 = 1$ with $|x| > 1/\sqrt{2}$.

7.3.1 Let t be the parameter for $\boldsymbol{\gamma}$, let s be arc-length along $\boldsymbol{\gamma}$, and denote d/dt by a dot and d/ds by a dash. Then, $\dot{\boldsymbol{\gamma}} = \frac{ds}{dt}\boldsymbol{\gamma}'$, $\ddot{\boldsymbol{\gamma}} = \left(\frac{ds}{dt}\right)^2\boldsymbol{\gamma}'' + \frac{d^2s}{dt^2}\boldsymbol{\gamma}'$. By Proposition 7.3.5, $\kappa_n = \langle\langle\boldsymbol{\gamma}',\boldsymbol{\gamma}'\rangle\rangle = \langle\langle\frac{1}{(ds/dt)}\dot{\boldsymbol{\gamma}}, \frac{1}{(ds/dt)}\dot{\boldsymbol{\gamma}}\rangle\rangle = \langle\langle\dot{\boldsymbol{\gamma}},\dot{\boldsymbol{\gamma}}\rangle\rangle/(ds/dt)^2 = \langle\langle\dot{\boldsymbol{\gamma}},\dot{\boldsymbol{\gamma}}\rangle\rangle/\langle\dot{\boldsymbol{\gamma}},\dot{\boldsymbol{\gamma}}\rangle$. For the second part, since $\boldsymbol{\gamma}'.(\mathbf{N}\times\boldsymbol{\gamma}') = \mathbf{0}$, we have $\ddot{\boldsymbol{\gamma}}.(\mathbf{N}\times\dot{\boldsymbol{\gamma}}) = \left(\frac{ds}{dt}\right)^3\boldsymbol{\gamma}''.(\mathbf{N}\times\boldsymbol{\gamma}') = \langle\dot{\boldsymbol{\gamma}},\dot{\boldsymbol{\gamma}}\rangle^{3/2}\kappa_g$.

7.3.2 Let $\boldsymbol{\gamma}$ be a unit-speed curve on the sphere of centre $\mathbf{a}$ and radius r. Then, $(\boldsymbol{\gamma}-\mathbf{a}).(\boldsymbol{\gamma}-\mathbf{a}) = r^2$; differentiating gives $\dot{\boldsymbol{\gamma}}.(\boldsymbol{\gamma}-\mathbf{a}) = 0$, so $\ddot{\boldsymbol{\gamma}}.(\boldsymbol{\gamma}-\mathbf{a}) = -\dot{\boldsymbol{\gamma}}.\dot{\boldsymbol{\gamma}} = -1$. At the point $\boldsymbol{\gamma}(t)$, the unit normal of the sphere is $\mathbf{N} = \pm\frac{1}{r}(\boldsymbol{\gamma}(t)-\mathbf{a})$, so $\kappa_n = \ddot{\boldsymbol{\gamma}}.\mathbf{N} = \pm\frac{1}{r}\ddot{\boldsymbol{\gamma}}.(\boldsymbol{\gamma}-\mathbf{a}) = \mp\frac{1}{r}$.

7.3.3 If the sphere has radius R, the parallel with latitude θ has radius $r = R\cos\theta$; if $\mathbf{p}$ is a point of this circle, its principal normal at $\mathbf{p}$ is parallel to the line through $\mathbf{p}$ perpendicular to the z-axis, while the unit normal to the sphere is parallel to the line through $\mathbf{p}$ and the centre of the sphere. The angle ψ in Eq. 7.10 is therefore equal to θ or $\pi - \theta$ so $\kappa_g = \pm\frac{1}{r}\sin\theta = \pm\frac{1}{R}\tan\theta$. Note that this is zero if and only if the parallel is a great circle.

7.3.4 We have $\dot{\boldsymbol{\gamma}} = \dot{u}\boldsymbol{\sigma}_u + \dot{v}\boldsymbol{\sigma}_v$, so by Exercise 6.4.7, $\mathbf{N}\times\dot{\boldsymbol{\gamma}} = \frac{\dot{u}(E\boldsymbol{\sigma}_v - F\boldsymbol{\sigma}_u)+\dot{v}(F\boldsymbol{\sigma}_v - G\boldsymbol{\sigma}_u)}{\sqrt{EG-F^2}}$, $\ddot{\boldsymbol{\gamma}} = \ddot{u}\boldsymbol{\sigma}_u + \ddot{v}\boldsymbol{\sigma}_v + \dot{u}^2\boldsymbol{\sigma}_{uu} + 2\dot{u}\dot{v}\boldsymbol{\sigma}_{uv} + \dot{v}^2\boldsymbol{\sigma}_{vv}$. Hence,

$$\kappa_g = \ddot{\boldsymbol{\gamma}}.(\mathbf{N}\times\dot{\boldsymbol{\gamma}}) = (\dot{u}\ddot{v} - \dot{v}\ddot{u})\sqrt{EG-F^2} + A\dot{u}^3 + B\dot{u}^2\dot{v} + C\dot{u}\dot{v}^2 + D\dot{v}^3,$$

where

$$\begin{aligned} A = \boldsymbol{\sigma}_{uu}.(E\boldsymbol{\sigma}_v - F\boldsymbol{\sigma}_u) &= E((\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_v)_u - \boldsymbol{\sigma}_u.\boldsymbol{\sigma}_{uv}) - \frac{1}{2}F(\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_u)_u \\ &= E(F_u - \frac{1}{2}E_v) - \frac{1}{2}FE_u, \end{aligned}$$

with similar expressions for B, C, D.

If $F = 0$, we find by this method that $A = -\frac{1}{2}E_v\sqrt{E/G}$, $B = G_u\sqrt{E/G} - \frac{1}{2}E_u\sqrt{G/E}$, $C = \frac{1}{2}G_v\sqrt{E/G} - E_v\sqrt{G/E}$, $D = \frac{1}{2}G_u\sqrt{G/E}$.

7.3.5 $\kappa_1 = \kappa\mathbf{N}_1.\mathbf{n}$, $\kappa_2 = \kappa\mathbf{N}_2.\mathbf{n}$, so

$$\kappa_1\mathbf{N}_2 - \kappa_2\mathbf{N}_1 = \kappa((\mathbf{N}_1.\mathbf{n})\mathbf{N}_2 - (\mathbf{N}_2.\mathbf{n})\mathbf{N}_1) = \kappa(\mathbf{N}_1\times\mathbf{N}_2)\times\mathbf{n}.$$

Taking the squared length of each side, we get

$$\kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2\mathbf{N}_1.\mathbf{N}_2 = \kappa^2 \parallel (\mathbf{N}_1\times\mathbf{N}_2)\times\mathbf{n} \parallel^2 .$$

Now, $\mathbf{N}_1.\mathbf{N}_2 = \cos\alpha$; $\dot{\boldsymbol{\gamma}}$ is perpendicular to $\mathbf{N}_1$ and $\mathbf{N}_2$, so $\mathbf{N}_1\times\mathbf{N}_2$ is parallel to $\dot{\boldsymbol{\gamma}}$, hence perpendicular to $\mathbf{n}$; hence, $\parallel (\mathbf{N}_1\times\mathbf{N}_2)\times\mathbf{n}\parallel = \parallel \mathbf{N}_1\times\mathbf{N}_2\parallel\parallel\mathbf{n}\parallel = \sin\alpha$.

7.3.6 A straight line has a unit-speed parametrization $\boldsymbol{\gamma}(t) = \mathbf{p} + t\,\mathbf{q}$ (with $\mathbf{q}$ a unit vector), so $\ddot{\boldsymbol{\gamma}} = \mathbf{0}$ and hence $\kappa_n = \ddot{\boldsymbol{\gamma}}.\mathbf{N} = 0$. In general, $\kappa_n = 0 \iff \ddot{\boldsymbol{\gamma}}$ is perpendicular to $\mathbf{N} \iff \mathbf{N}$ is perpendicular to $\mathbf{n} \iff \mathbf{N}$ is parallel to $\mathbf{b}$ (since $\mathbf{N}$ is perpendicular to $\mathbf{t}$).

7.3.7 The second fundamental form is $(-du^2 + u^2\,dv^2)/u\sqrt{1+u^2}$, so a curve $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ is asymptotic if and only if $-\dot{u}^2 + u^2\dot{v}^2 = 0$, i.e. $dv/du = \dot{v}/\dot{u} = \pm 1/u$, so $\ln u = \pm(v + c)$, where c is a constant.

7.3.8 By Proposition 7.3.2, if $\kappa_n = \kappa_g = 0$ then $\kappa = 0$. But a curve with zero curvature everywhere is part of a straight line.

7.3.9 We find that, at the point where $u = v = 1$, the first and second fundamental forms are $\frac{3}{2}du^2 + 2dudv + \frac{3}{2}dv^2$ and $\frac{2dudv}{\sqrt{5}}$, respectively. Taking $u = v = t$, so that $\dot{u} = \dot{v} = 1$, Exercise 7.3.1 gives the normal curvature as

$$\frac{\frac{2\dot{u}\dot{v}}{\sqrt{5}}}{\frac{3}{2}\dot{u}^2 + 2\dot{u}\dot{v} + \frac{3}{2}\dot{v}^2} = \frac{2}{5\sqrt{5}}.$$

7.3.10 From Examples 6.1.3 and 7.1.2, the first and second fundamental forms are $du^2 + f(u)^2dv^2$ and $(\dot{f}\ddot{g} - \ddot{f}\dot{g})du^2 + f\dot{g}dv^2$, respectively (a dot denoting d/du. We use the formulas in Exercise 7.3.1.
(i) For a meridian we use u as the parameter, so the curve is $\boldsymbol{\gamma}(u) = \boldsymbol{\sigma}(u, v)$ (with v constant). Note that $\boldsymbol{\gamma}$ is unit-speed. The second formula in Exercise 7.3.1 gives $\kappa_n = \dot{f}\ddot{g} - \ddot{f}\dot{g}$. Now, $\dot{\boldsymbol{\gamma}} = (\dot{f}\cos v, \dot{f}\sin v, \dot{g})$, $\ddot{\boldsymbol{\gamma}} = (\ddot{f}\cos v, \ddot{f}\sin v, \ddot{g})$, $\mathbf{N} = (-\dot{g}\cos v, -\dot{g}\sin v, \dot{f})$ (from Example 7.1.2 again), giving $\kappa_g = \ddot{\boldsymbol{\gamma}}.(\mathbf{N}\times\dot{\boldsymbol{\gamma}}) = 0$.
(ii) For a parallel we use v as the parameter, so the curve is $\boldsymbol{\Gamma}(v) = \boldsymbol{\sigma}(u, v)$ (with u constant). Note that $\boldsymbol{\Gamma}$ is not unit-speed in general. Now, $d\boldsymbol{\Gamma}/dv = (-f\sin v, f\cos v, 0)$, $d^2\boldsymbol{\Gamma}/dv^2 = (-f\cos v, -f\sin v, 0)$, giving $\ddot{\boldsymbol{\Gamma}}.(\mathbf{N}\times\dot{\boldsymbol{\Gamma}}) = f^2\dot{f}$. Hence, $\kappa_g = \frac{f^2\dot{f}}{f^3} = \frac{\dot{f}}{f}$.

7.3.11 The paraboloid $z = x^2 + y^2$ is obtained by rotating the parabola $z = x^2$ in the xz-plane around the z-axis. We could therefore use the preceding exercise but we would need a unit-speed parametrization of the parabola, which is complicated.
Proceeding directly, parametrize the paraboloid by $\boldsymbol{\sigma}(u, v) = (u\cos v, u\sin v, u^2)$. The first and second fundamental forms are $(1+4u^2)du^2+u^2dv^2$ and $\frac{2du^2+2u^2dv^2}{\sqrt{1+4u^2}}$, respectively. A curve $u =$ constant can be parametrized by $\boldsymbol{\gamma}(v) = \boldsymbol{\sigma}(u, v)$. Hence, $\kappa_n = \frac{\frac{2u^2}{\sqrt{1+4u^2}}}{u^2}$.
Now, denoting d/dv by a dot, we find that $\dot{\boldsymbol{\gamma}} = (-u\sin v, u\cos v, 0)$, $\ddot{\boldsymbol{\gamma}} = (-u\cos v, -u\sin v, 0)$, and $\mathbf{N} = \frac{1}{\sqrt{1+4u^2}}(-2u\cos v, -2u\sin v, 1)$, which gives $\ddot{\boldsymbol{\gamma}}.(\mathbf{N}\times\dot{\boldsymbol{\gamma}}) = \frac{u^2}{\sqrt{1+4u^2}}$. Hence, $\kappa_g = \frac{1}{u^3}\times\frac{u^2}{\sqrt{1+4u^2}} = \frac{1}{u\sqrt{1+4u^2}}$.

7.3.12 $\boldsymbol{\sigma}_u = \mathbf{t} + v\dot{\boldsymbol{\delta}}$, $\boldsymbol{\sigma}_v = \boldsymbol{\delta}$, so along $\boldsymbol{\gamma}$ (where $v = 0$) we have $\boldsymbol{\sigma}_u = \mathbf{t}$, $\boldsymbol{\sigma}_v = \boldsymbol{\delta}$ and hence $\mathbf{N} = \frac{\mathbf{t}\times\boldsymbol{\delta}}{\sin\theta}$. Hence,

$$\mathbf{N}\times\mathbf{t} = \frac{(\mathbf{t}\times\boldsymbol{\delta})\times\mathbf{t}}{\sin\theta} = \frac{\boldsymbol{\delta} - \cos\theta\mathbf{t}}{\sin\theta},$$

so

$$\kappa_g = \dot{\mathbf{t}}.(\mathbf{N}\times\mathbf{t}) = \frac{\dot{\mathbf{t}}.\boldsymbol{\delta}}{\sin\theta},$$

since $\dot{\mathbf{t}}.\mathbf{t} = 0$. Differentiating $\mathbf{t}.\boldsymbol{\delta} = \cos\theta$ gives $\dot{\mathbf{t}}.\boldsymbol{\delta} = -\sin\theta\dot{\theta} - \mathbf{t}.\dot{\boldsymbol{\delta}}$, hence the stated formula for κ_g.

7.3.13 A curve $v =$ constant is parametrized by $\boldsymbol{\gamma}(u) = \boldsymbol{\sigma}(u, v)$, so (denoting d/du by a dot) $\dot{\boldsymbol{\gamma}} = \boldsymbol{\sigma}_u$, $\ddot{\boldsymbol{\gamma}} = \boldsymbol{\sigma}_{uu}$ and $\kappa_g' = \boldsymbol{\sigma}_{uu}.(\mathbf{N} \times \boldsymbol{\sigma}_u)$. Now, $\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_u = \boldsymbol{\sigma}_v.\boldsymbol{\sigma}_v = 1$, $\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_v = \cos\theta$, so $\mathbf{N} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\sin\theta}$ and

$$\mathbf{N} \times \boldsymbol{\sigma}_u = \frac{1}{\sin\theta}(\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) \times \boldsymbol{\sigma}_u = \frac{1}{\sin\theta}(\boldsymbol{\sigma}_v - \cos\theta\boldsymbol{\sigma}_u).$$

Hence,

$$\kappa_g' = \frac{\boldsymbol{\sigma}_{uu}.\boldsymbol{\sigma}_v - \cos\theta\boldsymbol{\sigma}_{uu}.\boldsymbol{\sigma}_u}{\sin\theta}.$$

By Exercise 6.1.9, $\boldsymbol{\sigma}_{uu}.\boldsymbol{\sigma}_u = 0$, $\boldsymbol{\sigma}_{uu}.\boldsymbol{\sigma}_v = -\theta_u \sin\theta$. Inserting these formulas in the preceding equation for κ_g' gives $\kappa_g' = -\theta_u$. The formula for κ_g'' is proved similarly.

7.3.14 From the solution to Exercise 7.3.4,

$$\kappa_g = \sqrt{EG}(\dot{u}\ddot{v} - \ddot{u}\dot{v}) + \frac{X}{\sqrt{EG}},$$

where

$$\begin{aligned} X &= -\frac{1}{2}EE_v\dot{u}^3 + \left(EG_u - \frac{1}{2}GE_u\right)\dot{u}^2\dot{v} + \left(\frac{1}{2}EG_v - GE_v\right)\dot{u}\dot{v}^2 + \frac{1}{2}GG_u\dot{v}^3 \\ &= -\frac{1}{2}E_v(1 - G\dot{v}^2)\dot{u} + \left(EG_u - \frac{1}{2}GE_u\right)\dot{u}^2\dot{v} \\ &\qquad + \left(\frac{1}{2}EG_v - GE_v\right)\dot{u}\dot{v}^2 + \frac{1}{2}G_u(1 - E\dot{u}^2)\dot{v} \\ &= \frac{1}{2}(G_u\dot{v} - E_v\dot{u}) - \frac{1}{2}G(E_u\dot{u} + E_v\dot{v})\dot{u}\dot{v} + \frac{1}{2}E(G_u\dot{u} + G_v\dot{v})\dot{u}\dot{v}. \end{aligned}$$

Substituting into the formula for κ_g gives the desired result.

7.3.15 A parameter curve $v =$ constant can be parametrized by $\boldsymbol{\gamma}(u) = \boldsymbol{\sigma}(u, v)$, but this curve will not be unit-speed in general. Let s be the arc-length of $\boldsymbol{\gamma}$ and denote d/ds by a dot. Then, $ds/du = \parallel \boldsymbol{\sigma}_u \parallel = \sqrt{E}$, so $\dot{u} = 1/\sqrt{E}$. Since $\dot{v} = \ddot{v} = 0$, the formula in the preceding exercise gives

$$\kappa_g' = -\frac{E_v\dot{u}}{2\sqrt{EG}} = -\frac{E_v}{2E\sqrt{G}}.$$

The formula for κ_g'' is proved similarly.

We have $\cos\theta = \dot{u}\sqrt{E}$, $\sin\theta = \dot{v}\sqrt{G}$, so

$$\begin{aligned} \dot{\theta} &= \cos\theta\frac{d}{ds}(\sin\theta) - \sin\theta\frac{d}{ds}(\cos\theta) \\ &= \dot{u}\sqrt{E}\left(\ddot{v}\sqrt{G} + \frac{\dot{v}}{2\sqrt{G}}(G_u\dot{u} + G_v\dot{v})\right) - \dot{v}\sqrt{G}\left(\ddot{u}\sqrt{E} + \frac{\dot{u}}{2\sqrt{E}}(E_u\dot{u} + E_v\dot{v})\right) \\ &= \sqrt{EG}(\dot{u}\ddot{v} - \ddot{u}\dot{v}) + \frac{1}{2}\dot{u}\dot{v}\left(\sqrt{\frac{E}{G}}(G_u\dot{u} + G_v\dot{v}) - \sqrt{\frac{G}{E}}(E_u\dot{u} + E_v\dot{v})\right). \end{aligned}$$

Using the preceding exercise and the first part of this exercise, we get

$$\dot{\theta} = \kappa_g - \frac{1}{2\sqrt{EG}}(G_u\dot{v} - E_v\dot{u}) = \kappa_g + \kappa_g'\dot{u}\sqrt{E} + \kappa_g''\dot{v}\sqrt{G} = \kappa_g - \kappa_g'\cos\theta - \kappa_g''\sin\theta.$$

7.3.16 Let $\boldsymbol{\gamma}(t)$ be a unit-speed parametrization of $\mathcal{C}$, and let $\mathbf{p} = \boldsymbol{\gamma}(t_0)$. Let $\mathbf{N}$ be a unit normal of $\mathcal{S}$ at $\mathbf{p}$. Then, a parametrization of $\tilde{\mathcal{C}}$ is $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} - (\boldsymbol{\gamma}.\mathbf{N})\mathbf{N}$. Denoting d/dt by a dot as usual, $\dot{\tilde{\boldsymbol{\gamma}}} = \dot{\boldsymbol{\gamma}} - (\dot{\boldsymbol{\gamma}}.\mathbf{N})\mathbf{N}$, $\ddot{\tilde{\boldsymbol{\gamma}}} = \ddot{\boldsymbol{\gamma}} - (\ddot{\boldsymbol{\gamma}}.\mathbf{N})\mathbf{N}$, so the curvature of $\tilde{\boldsymbol{\gamma}}$ at $\mathbf{p}$ is

$$\tilde{\kappa} = \parallel \dot{\boldsymbol{\gamma}}(t_0) \times (\ddot{\boldsymbol{\gamma}}(t_0) - (\ddot{\boldsymbol{\gamma}}(t_0).\mathbf{N})\mathbf{N}) \parallel = \parallel \dot{\boldsymbol{\gamma}}(t_0) \times (\mathbf{N} \times (\ddot{\boldsymbol{\gamma}}(t_0) \times \mathbf{N})) \parallel,$$

since $\dot{\tilde{\boldsymbol{\gamma}}}(t_0) = \dot{\boldsymbol{\gamma}}(t_0)$ is a unit vector. Now, $\ddot{\boldsymbol{\gamma}}(t_0) = \kappa_g\mathbf{N} \times \dot{\boldsymbol{\gamma}}(t_0) + \kappa_n\mathbf{N}$, where κ_g and κ_n are the geodesic and normal curvatures of $\boldsymbol{\gamma}$ at $\mathbf{p}$. Hence, $\ddot{\boldsymbol{\gamma}}(t_0) \times \mathbf{N} = \kappa_g(\mathbf{N} \times \dot{\boldsymbol{\gamma}}(t_0)) \times \mathbf{N} = \kappa_g\dot{\boldsymbol{\gamma}}(t_0)$, so

$$\tilde{\kappa} = \parallel \dot{\boldsymbol{\gamma}}(t_0) \times (\kappa_g\mathbf{N} \times \dot{\boldsymbol{\gamma}}(t_0)) \parallel = |\kappa_g| \parallel \dot{\boldsymbol{\gamma}}(t_0) \parallel\parallel \mathbf{N} \times \dot{\boldsymbol{\gamma}}(t_0) \parallel = |\kappa_g|$$

as $\mathbf{N}$ and $\dot{\boldsymbol{\gamma}}(t_0)$ are perpendicular unit vectors.

7.3.17 The second fundamental form is $\frac{du^2 - dv^2}{\sqrt{1+u^2+v^2}}$, so a curve $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ is asymptotic if and only if $\dot{u}^2 - \dot{v}^2 = 0$, i.e. $u+v$ or $u-v$ is constant. If $u+v = c$, say, where c is a constant, then $\boldsymbol{\sigma}(u,v) = (u, c-u, \frac{1}{2}c(2u-c))$, which is a parametrization of a straight line. Similarly if $u-v$ is constant.

Geometrically, the surface is a hyperbolic paraboloid and is doubly ruled (Exercise 5.2.5). Both of the straight lines on the surface passing through a given point are asymptotic curves (Exercise 7.3.6).

7.3.18 In the usual notation, $\boldsymbol{\sigma}_u = \mathbf{t} + v(-\kappa\mathbf{t} + \tau\mathbf{b}) = \mathbf{t}$ along $\boldsymbol{\gamma}$; $\boldsymbol{\sigma}_v = \mathbf{n}$. Hence, along $\boldsymbol{\gamma}$, $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = \mathbf{b}$. Next, we find that $\boldsymbol{\sigma}_{uu} = -v\dot{\kappa}\mathbf{t} + (\kappa - (\kappa^2+\tau^2)v)\mathbf{n} + v\dot{\tau}\mathbf{b} = \kappa\mathbf{n}$ along $\boldsymbol{\gamma}$. The parameter curve $v = 0$ is asymptotic if and only if $L = 0$. But, along $\boldsymbol{\gamma}$, $L = \boldsymbol{\sigma}_{uu}.\mathbf{N} = \kappa\mathbf{n}.\mathbf{b} = 0$.

7.3.19 Parametrize the surface by $\boldsymbol{\sigma}(u,v) = (u^2\cos v, u^2\sin v, 2u)$. The second fundamental form is $\frac{-2du^2+2u^2dv^2}{\sqrt{1+u^2}}$, so the asymptotic curves are given by

$$-2\dot{u}^2 + 2u^2\dot{v}^2 = 0.$$

This gives $v = \pm\ln u + c$, where c is an arbitrary constant. The projection of this asymptotic curve on the xy-plane is given by $x = u^2\cos v$, $y = u^2\sin v$. Since $u^2 = e^{\pm 2(v-c)}$, this is a logarithmic spiral.

7.3.20 Let $\boldsymbol{\gamma}$ be a unit-speed parametrization of $\mathcal{C}$. Then, $\boldsymbol{\gamma}$ is asymptotic $\iff \ddot{\boldsymbol{\gamma}}.\mathbf{N} = 0 \iff \mathbf{n}.\mathbf{N} = 0$, where $\mathbf{N}$ is a unit normal to $\mathcal{S}$ and $\mathbf{n}$ is the principal normal of

$\boldsymbol{\gamma}$. Since $\dot{\boldsymbol{\gamma}} = \mathbf{t}$ is perpendicular to $\mathbf{N}$, $\boldsymbol{\gamma}$ is asymptotic $\Longleftrightarrow$ $\mathbf{N}$ is perpendicular to both $\mathbf{t}$ and $\mathbf{n}$. But the osculating plane is spanned by $\mathbf{t}$ and $\mathbf{n}$.

7.3.21 If every curve on a surface is asymptotic, $L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2 = 0$ for all curves $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ on the surface. This is possible only if $L = M = N = 0$ everywhere. By Exercise 7.1.2 this implies the surface is an open subset of a plane.

7.3.22 $\mathbf{N.n} = \cos\psi$, $\mathbf{N.t} = 0$, so $\mathbf{N.b} = \sin\psi$; hence, $\mathbf{N} = \mathbf{n}\cos\psi + \mathbf{b}\sin\psi$ and $\mathbf{B} = \mathbf{t} \times (\mathbf{n}\cos\psi + \mathbf{b}\sin\psi) = \mathbf{b}\cos\psi - \mathbf{n}\sin\psi$. Hence,

$$\begin{aligned}\dot{\mathbf{N}} &= \dot{\mathbf{n}}\cos\psi + \dot{\mathbf{b}}\sin\psi + \dot{\psi}(-\mathbf{n}\sin\psi + \mathbf{b}\cos\psi)\\ &= (-\kappa\mathbf{t} + \tau\mathbf{b})\cos\psi - \mathbf{n}\tau\sin\psi + \dot{\psi}(-\mathbf{n}\sin\psi + \mathbf{b}\cos\psi)\\ &= -\kappa\cos\psi\mathbf{t} + (\tau + \dot{\psi})(\mathbf{b}\cos\psi - \mathbf{n}\sin\psi)\\ &= -\kappa_n\mathbf{t} + \tau_g\mathbf{B}.\end{aligned}$$

The formula for $\dot{\mathbf{B}}$ is proved similarly. Since $\{\mathbf{t}, \mathbf{N}, \mathbf{B}\}$ is a right-handed orthonormal basis of $\mathbb{R}^3$, Exercise 2.3.6 shows that the matrix expressing $\dot{\mathbf{t}}, \dot{\mathbf{N}}, \dot{\mathbf{B}}$ in terms of $\mathbf{t}, \mathbf{N}, \mathbf{B}$ is skew-symmetric, hence the formula for $\dot{\mathbf{t}}$.

7.3.23 By Exercise 7.3.6, $\mathbf{b}$ is parallel to $\mathbf{N}$, so $\mathbf{b} = \pm\mathbf{N}$; then, $\mathbf{B} = \mathbf{t} \times \mathbf{N} = \mp\mathbf{n}$. Hence, $\dot{\mathbf{B}} = \mp\dot{\mathbf{n}} = \mp(-\kappa\mathbf{t} + \tau\mathbf{b}) = \pm\kappa\mathbf{t} - \tau\mathbf{N}$; comparing with the formula for $\dot{\mathbf{B}}$ in the preceding exercise shows that $\tau_g = \tau$ (and $\kappa_n = \pm\kappa$).

7.4.1 $\tilde{\mathbf{v}}$ is a smooth function of t and lies in $T_{\boldsymbol{\gamma}(\varphi(t))}\mathcal{S} = T_{\tilde{\boldsymbol{\gamma}}(t)}\mathcal{S}$, so $\tilde{\mathbf{v}}$ is a tangent vector field along $\tilde{\boldsymbol{\gamma}}$. The formula follows from Eq. (7.11) and the fact that $\frac{d\mathbf{v}}{dt} = \frac{d\tilde{\mathbf{v}}}{d\tilde{t}}\frac{d\varphi}{dt}$, where $\tilde{t} = \varphi(t)$. The last part follows since $\dot{\varphi} \neq 0$ so $\nabla_{\tilde{\boldsymbol{\gamma}}}\tilde{\mathbf{v}} = \mathbf{0} \Longleftrightarrow \nabla_{\boldsymbol{\gamma}}\mathbf{v} = \mathbf{0}$.

7.4.2 If $\mathbf{p}$ and $\mathbf{q}$ correspond to the parameter values $t = a$ and $t = b$, respectively, let $\boldsymbol{\Gamma}(t) = \boldsymbol{\gamma}(a + b - t)$ (thus, $\boldsymbol{\Gamma}$ is $\boldsymbol{\gamma}$ 'traversed backwards'). We show that $\Pi_{\boldsymbol{\Gamma}}^{\mathbf{qp}}$ is the inverse of $\Pi_{\boldsymbol{\gamma}}^{\mathbf{pq}}$. Let $\mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$ and let $\mathbf{v}$ be the tangent vector field parallel along $\boldsymbol{\gamma}$ such that $\mathbf{v}(a) = \mathbf{w}$. Then, $\Pi_{\boldsymbol{\gamma}}^{\mathbf{pq}}(\mathbf{w}) = \mathbf{v}(b)$. By Exercise 7.4.1, $\mathbf{V}(t) = \mathbf{v}(a + b - t)$ is parallel along $\boldsymbol{\Gamma}$ so $\Pi_{\boldsymbol{\Gamma}}^{\mathbf{qp}}(\mathbf{v}(b)) = \Pi_{\boldsymbol{\Gamma}}^{\mathbf{qp}}(\mathbf{V}(a)) = \mathbf{V}(b) = \mathbf{v}(a) = \mathbf{w}$. This proves that $\Pi_{\boldsymbol{\Gamma}}^{\mathbf{qp}} \circ \Pi_{\boldsymbol{\gamma}}^{\mathbf{pq}}$ is the identity map on $T_{\mathbf{p}}\mathcal{S}$. One proves similarly (or by interchanging the roles of $\boldsymbol{\gamma}$ and $\boldsymbol{\Gamma}$) that $\Pi_{\boldsymbol{\gamma}}^{\mathbf{pq}} \circ \Pi_{\boldsymbol{\Gamma}}^{\mathbf{qp}}$ is the identity map on $T_{\mathbf{q}}\mathcal{S}$.

7.4.3 Let α, β, γ be the internal angles of the triangle at $\mathbf{p}, \mathbf{q}, \mathbf{r}$, respectively. Since the arc through $\mathbf{p}$ and $\mathbf{q}$ is part of a great circle, the tangent vector of the arc is parallel along the arc (Example 7.4.7). So the result of parallel transporting $\mathbf{v}_0$ to $\mathbf{q}$ along the arc $\overline{\mathbf{pq}}$ through $\mathbf{p}$ and $\mathbf{q}$ is a vector $\mathbf{v}_1$ tangent to $\overline{\mathbf{pq}}$ at $\mathbf{q}$. Now $\mathbf{v}_1$ makes an angle $\pi - \beta$ with the arc $\overline{\mathbf{qr}}$ at $\mathbf{q}$, so parallel transporting $\mathbf{v}_1$ along $\overline{\mathbf{qr}}$

to $\mathbf{r}$ gives a vector $\mathbf{v}_2$ which makes an angle $(\pi-\beta)+(\pi-\gamma)$ with the arc $\overline{\mathbf{rp}}$ at $\mathbf{r}$. Parallel transporting $\mathbf{v}_2$ along $\overline{\mathbf{rp}}$ to $\mathbf{p}$ then gives a vector $\mathbf{v}_3$ which makes an angle $(\pi-\beta)+(\pi-\gamma)+(\pi-\alpha)$ with $\mathbf{v}_0$. Since $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ all have the same length (Proposition 7.4.9(ii)), the result follows from Theorem 6.4.7.

7.4.4 Putting $E=G=1$, $F=\cos\theta$ in the formulas in Proposition 7.4.4 gives $\Gamma^1_{11} = \theta_u \cot\theta$, $\Gamma^2_{11} = -\theta_u \operatorname{cosec}\theta$, $\Gamma^1_{22} = -\theta_v \operatorname{cosec}\theta$, $\Gamma^2_{22} = \theta_v\cot\theta$, $\Gamma^1_{12} = \Gamma^2_{12} = 0$.

7.4.5 In the notation of Proposition 7.4.5, the tangent vector $\boldsymbol{\sigma}_v$ to the parameter curves $u=$ constant are given by $\alpha=0$, $\beta=1$, and a curve $v=$ constant corresponds to $\dot v = 0$ (and $\dot u \neq 0$). From Eq. 7.13, $\boldsymbol{\sigma}_v$ is parallel along the curves $v=$ constant if and only if $\Gamma^1_{12} = \Gamma^2_{12} = 0$; from the formulas in Proposition 7.4.4, this holds if and only if

$$(*) \qquad GE_v = FG_u \text{ and } EG_u = FE_v.$$

If the parameter curves form a Chebyshev net, $E_v = G_u = 0$ (Exercise 6.1.5), so (*) holds. Conversely, if (*) holds than

$$(EG-F^2)G_u = G(EG_u) - F(FG_u) = G(FE_v) - F(GE_v) = 0$$

and so $G_u = 0$; then $GE_v = 0$ by (*) again, so $E_v = 0$. Hence, if (*) holds the parameter curves form a Chebyshev net. This proves that (i) is equivalent to (ii). That (i) is equivalent to (iii) is proved similarly.

7.4.6 We have $\cos\theta = \frac{\boldsymbol{\sigma}_u . \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u\|\|\boldsymbol{\sigma}_v\|} = F/\sqrt{EG}$, so $A = \sqrt{EG - EG\cos^2\theta} = \sqrt{EG}\sin\theta$. Differentiating the first equation with respect to u and using the second equation gives

$$-\theta_u = \frac{\sqrt{EG}}{A}\left(\frac{F}{\sqrt{EG}}\right)_u = \frac{2EGF_u - FGE_u - EFG_u}{2AEG}.$$

On the other hand, the formulas in Proposition 7.4.4 give

$$\begin{aligned}\frac{\Gamma^2_{11}}{E} + \frac{\Gamma^1_{12}}{G} &= \frac{1}{2A^2}\left(\frac{2EF_u - EE_v - FE_u}{E} + \frac{GE_v - FG_u}{G}\right) \\ &= \frac{2EGF_u - FGE_u - EFG_u}{2A^2EG}.\end{aligned}$$

7.4.7 From $A^2 = EG - F^2$ we get $2AA_u = EG_u + GE_u - 2FF_u$. On the other hand, the formulas in Proposition 7.4.4 give $\Gamma^1_{11} + \Gamma^2_{12} = \frac{GE_u + EG_u - 2FF_u}{2A^2}$. By the preceding formula, this is equal to $2AA_u/2A^2 = A_u/A$. The other formula is proved similarly.

Chapter 8

8.1.1 Parametrize the surface by $\boldsymbol{\sigma}(x,y) = (x, y, f(x,y))$. Then, $\boldsymbol{\sigma}_x = (1, 0, f_x)$, $\boldsymbol{\sigma}_y = (0, 1, f_y)$, $\mathbf{N} = (1 + f_x^2 + f_y^2)^{-1/2}(-f_x, -f_y, 1)$, $\boldsymbol{\sigma}_{xx} = (0, 0, f_{xx})$, $\boldsymbol{\sigma}_{xy} = (0, 0, f_{xy})$, $\boldsymbol{\sigma}_{yy} = (0, 0, f_{yy})$. This gives $E = 1 + f_x^2$, $F = f_x f_y$, $G = 1 + f_y^2$ and $L = (1 + f_x^2 + f_y^2)^{-1/2} f_{xx}$, $M = (1 + f_x^2 + f_y^2)^{-1/2} f_{xy}$, $N = (1 + f_x^2 + f_y^2)^{-1/2} f_{yy}$. By Corollary 8.1.3, $K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1+f_x^2+f_y^2)^2}$, $H = \frac{(1+f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1+f_x^2) f_{yy}}{2(1+f_x^2+f_y^2)^{3/2}}$.

8.1.2 For the helicoid $\boldsymbol{\sigma}(u,v) = (v\cos u, v\sin u, \lambda u)$,

$$\begin{aligned}
\boldsymbol{\sigma}_u &= (-v\sin u, v\cos u, \lambda),\\
\boldsymbol{\sigma}_v &= (\cos u, \sin u, 0),\\
\mathbf{N} &= \frac{1}{\sqrt{\lambda^2 + v^2}}(-\lambda \sin u, \lambda\cos u, -v),\\
\boldsymbol{\sigma}_{uu} &= (-v\cos u, -v\sin u, 0),\\
\boldsymbol{\sigma}_{uv} &= (-\sin u, \cos u, 0),\\
\boldsymbol{\sigma}_{vv} &= \mathbf{0}.
\end{aligned}$$

This gives $E = \lambda^2 + v^2, F = 0, G = 1$ and $L = N = 0, M = \lambda/\sqrt{\lambda^2 + v^2}$. Hence, $K = (LN - M^2)/(EG - F^2) = -\lambda^2/(\lambda^2 + v^2)^2$.

For the catenoid $\boldsymbol{\sigma}(u,v) = (\cosh u \cos v, \cosh u \sin v, u)$,

$$\begin{aligned}
\boldsymbol{\sigma}_u &= (\sinh u \cos v, \sinh u \sin v, 1),\\
\boldsymbol{\sigma}_v &= (-\cosh u \sin v, \cosh u \cos v, 0),\\
\mathbf{N} &= \operatorname{sech} u(-\cos v, -\sin v, \sinh u),\\
\boldsymbol{\sigma}_{uu} &= (\cosh u \cos v, \cosh u \sin v, 0),\\
\boldsymbol{\sigma}_{uv} &= (-\sinh u \sin v, \sinh u \cos v, 0),\\
\boldsymbol{\sigma}_{vv} &= (-\cosh u \cos v, -\cosh u \sin v, 0).
\end{aligned}$$

This gives $E = G = \cosh^2 u, F = 0$ and $L = -1, M = 0, N = 1$. Hence, $K = (LN - M^2)/(EG - F^2) = -\operatorname{sech}^4 u$.

8.1.3 Since $\boldsymbol{\sigma}$ is smooth and $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v$ is never zero, $\mathbf{N} = \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v / \parallel \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \parallel$ is smooth. Hence, E, F, G, L, M and N are smooth. Since $EG - F^2 > 0$ (by the remark following Proposition 6.4.2), the formulas in Corollary 8.1.3 show that H and K are smooth.

8.1.4 From Example 8.1.5, $K = 0 \iff \dot{\boldsymbol{\delta}}.\mathbf{N} = 0 \iff \dot{\boldsymbol{\delta}}.((\mathbf{t} + v\dot{\boldsymbol{\delta}}) \times \boldsymbol{\delta}) = 0 \iff \dot{\boldsymbol{\delta}}.(\mathbf{t} \times \boldsymbol{\delta}) = 0$. If $\boldsymbol{\delta} = \mathbf{n}$, $\dot{\boldsymbol{\delta}} = -\kappa \mathbf{t} + \tau \mathbf{b}$, $\mathbf{t} \times \boldsymbol{\delta} = \mathbf{b}$, so $K = 0 \iff \tau = 0 \iff \boldsymbol{\gamma}$ is planar (by Proposition 2.3.3). If $\boldsymbol{\delta} = \mathbf{b}$, $\dot{\boldsymbol{\delta}} = -\tau\mathbf{n}$, $\mathbf{t} \times \boldsymbol{\delta} = -\mathbf{n}$, so again $K = 0 \iff \tau = 0$.

8.1.5 The dilation $(x, y, z) \mapsto (ax, ay, az)$, where a is a non-zero constant, multiplies E, F, G by a^2 and L, M, N by a, hence H by a^{-1} and K by a^{-2} (using Corollary 8.1.3).

8.1.6 This follows immediately from Definition 8.1.1 and the hint.

8.1.7 Suppose that the cone is the union of the straight lines joining points of a curve $\mathcal{C}$ to a vertex $\mathbf{v}$. It is clear that the Gauss map $\mathcal{G}$ is constant along the rulings of the cone, so the image of the cone under $\mathcal{G}$ is the same as the image of $\mathcal{C}$ under $\mathcal{G}$, which is a curve.

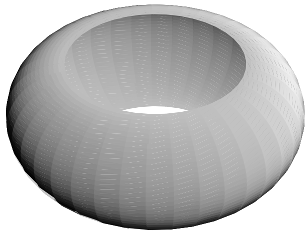
8.1.8 By Eq. 8.2, the area of $\boldsymbol{\sigma}(R)$ is

$$\int\!\!\int_R \parallel \mathbf{N}_u \times \mathbf{N}_v \parallel dudv = \int\!\!\int_R |K| \parallel \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \parallel dudv = \int\!\!\int_R |K| d\mathcal{A}_{\boldsymbol{\sigma}}.$$

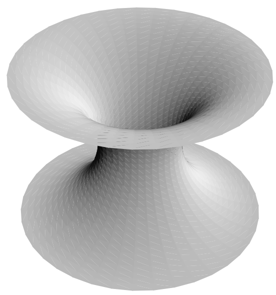
8.1.9 Using the parametrization $\boldsymbol{\sigma}$ in Exercise 4.2.5, we find that $E = b^2, F = 0, G = (a+b\cos\theta)^2$ and $L = b, M = 0, N = (a+b\cos\theta)\cos\theta$. This gives $K = \frac{\cos\theta}{b(a+b\cos\theta)}$. It follows that $\mathcal{S}^+$ and $\mathcal{S}^-$ are the annular regions on the torus given by

$$-\pi/2 \leq u \leq \pi/2 \text{ and } \pi/2 \leq u \leq 3\pi/2,$$

respectively.



$\mathcal{S}^+$



$\mathcal{S}^-$

It is clear that as a point $\mathbf{p}$ moves over $\mathcal{S}^+$ (resp. $\mathcal{S}^-$), the unit normal at $\mathbf{p}$ covers the whole of S^2. Hence, $\int\!\!\int_{\mathcal{S}^+} |K| d\mathcal{A} = \int\!\!\int_{\mathcal{S}^-} |K| d\mathcal{A} = 4\pi$ by the preceding exercise; since $|K| = \pm K$ on $\mathcal{S}^\pm$, this gives the result.

8.1.10 $\nabla_u \mathbf{w} = \mathbf{w}_u - (\mathbf{w}_u.\mathbf{N})\mathbf{N}$ so

$$\begin{aligned}\nabla_v(\nabla_u \mathbf{w}) &= \mathbf{w}_{uv} - (\mathbf{w}_{uv}.\mathbf{N})\mathbf{N} - (\mathbf{w}_u.\mathbf{N}_v)\mathbf{N} - (\mathbf{w}_u.\mathbf{N})\mathbf{N}_v - (\mathbf{w}_{uv}.\mathbf{N})\mathbf{N} \\ &\qquad + (\mathbf{w}_{uv}.\mathbf{N})\mathbf{N} + (\mathbf{w}_u.\mathbf{N}_v)\mathbf{N} + (\mathbf{w}_u.\mathbf{N})(\mathbf{N}_v.\mathbf{N})\mathbf{N} \\ &= \mathbf{w}_{uv} - (\mathbf{w}_{uv}.\mathbf{N})\mathbf{N} - (\mathbf{w}_u.\mathbf{N})\mathbf{N}_v.\end{aligned}$$

Interchanging u and v and subtracting gives the first formula. Replacing $\mathbf{w}$ by $\lambda\mathbf{w}$ in this formula gives

$$\lambda\{(\mathbf{w}_v.\mathbf{N})\mathbf{N}_u - (\mathbf{w}_u.\mathbf{N})\mathbf{N}_v\} + \lambda_v(\mathbf{w}.\mathbf{N})\mathbf{N}_u - \lambda_u(\mathbf{w}.\mathbf{N})\mathbf{N}_v = \lambda\{(\mathbf{w}_v.\mathbf{N})\mathbf{N}_u - (\mathbf{w}_u.\mathbf{N})\mathbf{N}_v\}$$

since $\mathbf{w}.\mathbf{N} = 0$. It is also obvious that

$$\begin{aligned}\nabla_v(\nabla_u(\mathbf{w}_1 + \mathbf{w}_2)) - \nabla_u(\nabla_v(\mathbf{w}_1 + \mathbf{w}_2)) = &(\nabla_v(\nabla_u\mathbf{w}_1) - \nabla_u(\nabla_v\mathbf{w}_1)) \\ &+ (\nabla_v(\nabla_u\mathbf{w}_2) - \nabla_u(\nabla_v\mathbf{w}_2))\end{aligned}$$

for any two tangent vector fields $\mathbf{w}_1, \mathbf{w}_2$.

Now $\nabla_v(\nabla_u\boldsymbol{\sigma}_u) - \nabla_u(\nabla_v\boldsymbol{\sigma}_u) = (\boldsymbol{\sigma}_{uv}.\mathbf{N})\mathbf{N}_u - (\boldsymbol{\sigma}_{uu}.\mathbf{N})\mathbf{N}_v = M\mathbf{N}_u - L\mathbf{N}_v$ (in the usual notation). Using Proposition 8.1.2, this $= M(a\boldsymbol{\sigma}_u + b\boldsymbol{\sigma}_v) - L(c\boldsymbol{\sigma}_u + d\boldsymbol{\sigma}_v)$ and using the explicit expressions for a, b, c, d in Proposition 8.1.2 this becomes $K(E\boldsymbol{\sigma}_v - F\boldsymbol{\sigma}_u)$. Similarly, $\nabla_v(\nabla_u\boldsymbol{\sigma}_v) - \nabla_u(\nabla_v\boldsymbol{\sigma}_v) = K(-G\boldsymbol{\sigma}_u + F\boldsymbol{\sigma}_v)$.

If

$$(*) \qquad \nabla_v(\nabla_u\mathbf{w}) - \nabla_u(\nabla_v\mathbf{w}) = \mathbf{0}$$

for all $\mathbf{w}$, then taking $\mathbf{w} = \boldsymbol{\sigma}_u$ gives $K = 0$ since $E \neq 0$, $\boldsymbol{\sigma}_v \neq \mathbf{0}$. Conversely, if $K = 0$ then (*) holds for $\mathbf{w} = \boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$, and hence by the first part of the exercise it holds for $\alpha\boldsymbol{\sigma}_u + \beta\boldsymbol{\sigma}_v$ for all smooth functions α, β of (u, v). But every tangent vector field $\mathbf{w}$ is of this form.

8.1.11 $\boldsymbol{\sigma}_u = (1, 1, v)$, $\boldsymbol{\sigma}_v = (1, -1, u)$, $\boldsymbol{\sigma}_{uu} = \boldsymbol{\sigma}_{vv} = \mathbf{0}$, $\boldsymbol{\sigma}_{uv} = (0, 0, 1)$. When $u = v = 1$, we find from this that $E = 3, F = 1, G = 3$ and $L = N = 0, M = -1/\sqrt{2}$. Hence, $K = (LN - M^2)/(EG - F^2) = -1/16$, $H = (LG - 2MF + NG)/2(EG - F^2) = 1/8\sqrt{2}$.

8.1.12 We write the equation of the quadric in the form $z = f(x, y)$, where $f(x, y) = \sqrt{c\left(1 - \frac{x^2}{a} - \frac{y^2}{b}\right)}$. The result of the calculation will give the value of the Gaussian curvature K on the part of the quadric where $z > 0$. Similar calculations (with the same result) give K on the parts of the surface where $z < 0$, $x > 0$, $x < 0$, $y > 0$ and $y < 0$. Together these regions cover the whole surface.

Taking $f(x, y) = \sqrt{c\left(1 - \frac{x^2}{a} - \frac{y^2}{b}\right)}$, and applying the first formula in Exercise 8.1.1, we get, after straightforward calculation,

$$K = \frac{1}{abc}\frac{\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c}}{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^2}.$$

The vector $\nabla\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=2\left(\frac{x}{a},\frac{y}{b},\frac{z}{c}\right)$ is normal to the surface, so

$$d=\left|\frac{(x,y,z).\left(\frac{x}{a},\frac{y}{b},\frac{z}{c}\right)}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}}\right|=\frac{1}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}}.$$

This gives the stated formula for K.

For the paraboloid we take $f(x,y)=\frac{1}{4}\left(\frac{x^2}{a}+\frac{y^2}{b}\right)$. The first formula in Exercise 8.1.1 gives

$$K=\frac{1}{4ab\left(1+\frac{x^2}{4a^2}+\frac{y^2}{4b^2}\right)^2}.$$

The vector $\left(\frac{x}{2a},\frac{y}{2b},-1\right)$ is normal to the surface, so

$$d=\left|\frac{(x,y,z).\left(\frac{x}{2a},\frac{y}{2b},-1\right)}{\sqrt{\frac{x^2}{4a^2}+\frac{y^2}{4b^2}+1}}\right|=\frac{\left|\frac{x^2}{2a}+\frac{y^2}{2b}-z\right|}{\sqrt{\frac{x^2}{4a^2}+\frac{y^2}{4b^2}+1}}=\frac{|z|}{\sqrt{\frac{x^2}{4a^2}+\frac{y^2}{4b^2}+1}}.$$

Hence, $K=d^4/4abz^4$.

8.1.13 Taking $f(x,y)=1/xy$, we get $f_x=-z/x$, $f_y=-z/y$, $f_{xx}=2z/x^2$, $f_{xy}=z/xy$, $f_{yy}=2z/y^2$. The first formula in Exercise 8.1.1 gives

$$K=\frac{\frac{4z^2}{x^2y^2}-\frac{z^2}{x^2y^2}}{\left(1+\frac{z^2}{x^2}+\frac{z^2}{y^2}\right)^2}=\frac{3x^{-2}y^{-2}z^{-2}}{(x^{-2}+y^{-2}+z^{-2})^2},$$

hence the stated formula for K. Similarly, the second formula in Exercise 8.1.1 gives

$$H=\frac{\frac{2z}{x^2}\left(1+\frac{z^2}{y^2}\right)-\frac{2z^3}{x^2y^2}+\frac{2z}{y^2}\left(1+\frac{z^2}{x^2}\right)}{\frac{2}{x^3y^3}(x^2+y^2+z^2)^{3/2}}=\frac{xyz(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}=\frac{1}{\sqrt{x^2+y^2+z^2}}.$$

We showed in Exercise 5.1.3 that the smallest value of $X^2+Y^2+Z^2$ subject to the condition $XYZ=1$ is 3, and that it is attained precisely when $(X,Y,Z)=(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$ or $(1,1,1)$. Taking $X=1/x$, $Y=1/y$, $Z=1/z$, it follows that the largest value of K is $3(1+1+1)^{-2}=1/3$, and that it is attained precisely when $(1/x,1/y,1/z)=(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$ or $(1,1,1)$, i.e. when $(x,y,z)=(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$ or $(1,1,1)$. As noted in the solution to Exercise 5.1.3, these points lie at the vertices of a regular tetrahedron.

8.1.14 If the circle in the xz-plane has radius a and touches the z-axis at the point $(0,0,c)$, it can be parametrized by $\boldsymbol{\gamma}(u) = (a(1+\cos u), 0, a\sin u + c)$. Rotating $\boldsymbol{\gamma}(u)$ around the z-axis through an angle v gives the point

$$(a(1+\cos u)\cos v, a(1+\cos u)\sin v, a\sin u + c).$$

Since the circle rotates at constant angular velocity and moves parallel to the z-axis at constant speed, the distance it moves parallel to the z-axis in the time taken to rotate through an angle v is a constant multiple of v, say bv. Hence, after the circle has rotated through an angle v the point initially at $\boldsymbol{\gamma}(u)$ will have moved to the point $\boldsymbol{\sigma}(u,v)$ in the statement of the exercise.

If $a = b$ and $c = 0$ we have $\boldsymbol{\sigma}(u,v) = a((1+\cos u)\cos v, (1+\cos u)\sin v, \sin u + v)$. We find that the first fundamental form is

$$a^2(du^2 + 2\cos u du dv + ((1+\cos u)^2 + 1)dv^2).$$

Let $A = \parallel \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \parallel = \sqrt{EG - F^2}$, so that $A^2 = 2a^4(1+\cos u)$. The second fundamental form is

$$\frac{a^3}{A}((1+\cos u)du^2 - 2\sin^2 u du dv + (1+\cos u)^2 \cos u dv^2).$$

The first formula in Corollary 8.1.3 now gives

$$K = \frac{a^6((1+\cos u)^3 \cos u - \sin^4 u)}{A^4}.$$

Writing $c = \cos u$ we get

$$K = \frac{c(1+c)^3 - (1-c^2)^2}{4a^2(1+c)^2} = \frac{c(1+c) - (1-c)^2}{4a^2} = \frac{3\cos u - 1}{4a^2}.$$

Since $d = a(1+\cos u)$, this agrees with the formula in the statement of the exercise.

8.1.15 Using the parametrization in Example 8.1.5, we have $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = \dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta} + v\dot{\boldsymbol{\delta}} \times \boldsymbol{\delta}$. Recalling that $\parallel \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \parallel = \sqrt{EG - F^2}$, the formula for K in that example becomes

$$K = \frac{-(\dot{\boldsymbol{\delta}}.(\dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta}))^2}{\parallel \dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta} + v\dot{\boldsymbol{\delta}} \times \boldsymbol{\delta} \parallel^4}.$$

If $\dot{\boldsymbol{\delta}} \times \boldsymbol{\delta} \neq \mathbf{0}$, $K \to 0$ as $v \to \infty$ (with u fixed); thus, if K is constant we must have $K = 0$. On the other hand, if $\dot{\boldsymbol{\delta}} \times \boldsymbol{\delta} = \mathbf{0}$, $\dot{\boldsymbol{\delta}}$ is parallel to $\boldsymbol{\delta}$, hence $\dot{\boldsymbol{\delta}}.(\dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta}) = 0$ and again $K = 0$.

8.1.16 We use the parametrization $\boldsymbol{\sigma}(s,\theta)$ in Exercise 4.2.7, a dot to denote d/ds, and the standard notation relating to the Frenet-Serret data for $\boldsymbol{\gamma}$. Then, $\boldsymbol{\sigma}_s = (1-\kappa a\cos\theta)\mathbf{t} - \tau a\sin\theta\mathbf{n} + \tau a\cos\theta\mathbf{b}$, $\boldsymbol{\sigma}_\theta = -a\sin\theta\mathbf{n} + a\cos\theta\mathbf{b}$. Hence,

$$\boldsymbol{\sigma}_s \times \boldsymbol{\sigma}_\theta = -a(1-\kappa a\cos\theta)(\cos\theta\mathbf{n} + \sin\theta\mathbf{b}),$$

so $\mathbf{N} = -(\cos\theta\mathbf{n} + \sin\theta\mathbf{b})$, $E = (1-\kappa a\cos\theta)^2 + \tau^2 a^2$, $F = \tau a^2$, $G = a^2$ and $EG - F^2 = a^2(1-\kappa a\cos\theta)^2$. Next,

$$\begin{aligned}
\boldsymbol{\sigma}_{ss} &= (\kappa\tau a\sin\theta - \dot\kappa a\cos\theta)\mathbf{t} + (\kappa(1-\kappa a\cos\theta) - \dot\tau a\sin\theta - \tau^2 a\cos\theta)\mathbf{n}\\
&\qquad + (\dot\tau a\cos\theta - \tau^2 a\sin\theta)\mathbf{b},\\
\boldsymbol{\sigma}_{s\theta} &= \kappa a\sin\theta\mathbf{t} - \tau a\cos\theta\mathbf{n} - \tau a\sin\theta\mathbf{b},\\
\boldsymbol{\sigma}_{\theta\theta} &= -a\cos\theta\mathbf{n} - a\sin\theta\mathbf{b}.
\end{aligned}$$

Hence, $L = a\tau^2 - \kappa\cos\theta(1-\kappa a\cos\theta)$, $M = \tau a$, $N = a$, Thus,

$$K = \frac{-\kappa a\cos\theta(1-\kappa a\cos\theta)}{a^2(1-\kappa a\cos\theta)^2} = \frac{-\kappa\cos\theta}{a(1-\kappa a\cos\theta)}.$$

(i) We have $d\mathcal{A} = \parallel \boldsymbol{\sigma}_s \times \boldsymbol{\sigma}_\theta \parallel\ dsd\theta = a(1-\kappa a\cos\theta)dsd\theta$, so

$$\int_0^\ell \int_0^{2\pi} K d\mathcal{A} = \int_0^\ell \int_0^{2\pi} (-\kappa\cos\theta)dsd\theta = 0.$$

(ii) Since $\int_0^{2\pi} |\cos\theta| d\theta = 2\int_{-\pi/2}^{\pi/2} \cos\theta d\theta = 4$, we get

$$\int_0^\ell \int_0^{2\pi} |K| d\mathcal{A} = 4\int_0^\ell \kappa(s)ds.$$

8.1.17 It follows from Exercises 6.1.2 and 7.1.4 that applying a direct isometry of $\mathbb{R}^3$ leaves both H and K unchanged, while applying an opposite isometry leaves K unchanged but changes the sign of H.

8.1.18 From the proof of Proposition 7.3.3, if $\boldsymbol{\gamma}$ is unit-speed its normal curvature is $\kappa_n = -\dot{\mathbf{N}}.\dot{\boldsymbol{\gamma}} = \mathcal{W}(\dot{\boldsymbol{\gamma}}).\dot{\boldsymbol{\gamma}}$. Hence, $\boldsymbol{\gamma}$ is asymptotic if and only if $\mathcal{W}(\dot{\boldsymbol{\gamma}})$ is perpendicular to $\dot{\boldsymbol{\gamma}}$ at each point of $\boldsymbol{\gamma}$. Since $-\mathcal{W}(\dot{\boldsymbol{\gamma}})$ is the tangent vector of the image of $\boldsymbol{\gamma}$ under the Gauss map of $\mathcal{S}$, the result follows.

8.1.19 Recalling that $\dot{\mathbf{N}} = -\mathcal{W}(\dot{\boldsymbol{\gamma}})$, and that $\mathcal{W}$ is self-adjoint (Corollary 7.2.4), we have to prove that

$$\langle \mathcal{W}^2(\dot{\boldsymbol{\gamma}}) - 2H\mathcal{W}(\dot{\boldsymbol{\gamma}}) + K\dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}}\rangle = 0.$$

This follows from the preceding exercise.

We can assume that $\boldsymbol{\gamma}$ is unit-speed. As in the solution of Exercise 8.1.18, if $\boldsymbol{\gamma}$ is asymptotic then $\dot{\mathbf{N}}.\dot{\boldsymbol{\gamma}} = 0$ and the first part gives $\dot{\mathbf{N}}.\dot{\mathbf{N}} = -K$. But by Exercise 7.3.6, if $\boldsymbol{\gamma}$ is asymptotic, $\mathbf{N} = \pm\mathbf{b}$ so $\dot{\mathbf{N}} = \mp\tau\mathbf{n}$ (where $\mathbf{n}$ is the principal normal of $\boldsymbol{\gamma}$ and $\mathbf{b}$ is its binormal). Hence, $\dot{\mathbf{N}}.\dot{\mathbf{N}} = \tau^2$.

8.2.1 For the helicoid $\boldsymbol{\sigma}(u, v) = (v\cos u, v\sin u, \lambda u)$, the first and second fundamental forms are $(\lambda^2 + v^2)du^2 + dv^2$ and $2\lambda du dv/\sqrt{\lambda^2 + v^2}$, respectively. Hence, the principal curvatures are the roots of $\begin{vmatrix} -\kappa(\lambda^2 + v^2) & \frac{\lambda}{\sqrt{\lambda^2+v^2}} \\ \frac{\lambda}{\sqrt{\lambda^2+v^2}} & -\kappa \end{vmatrix} = 0$, i.e. $\kappa = \pm\lambda/(\lambda^2 + v^2)$.

For the catenoid $\boldsymbol{\sigma}(u, v) = (\cosh u \cos v, \cosh u \sin v, u)$, the first and second fundamental forms are $\cosh^2 u(du^2+dv^2)$ and $-du^2+dv^2$, so the principal curvatures are the roots of $\begin{vmatrix} -1 - \kappa\cosh^2 u & 0 \\ 0 & 1 - \kappa\cosh^2 u \end{vmatrix} = 0$, i.e. $\kappa = \pm\mathrm{sech}^2 u$.

8.2.2 This is obvious, since $\mathcal{W}(\dot{\boldsymbol{\gamma}}) = -\dot{\mathbf{N}}$.

8.2.3 $\boldsymbol{\gamma}$ is a line of curvature $\iff \dot{\boldsymbol{\gamma}} = \dot{u}\boldsymbol{\sigma}_u + \dot{v}\boldsymbol{\sigma}_n$ is a principal vector for all $t \iff$ $\begin{pmatrix} L & M \\ L & N \end{pmatrix} = \kappa \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ for some scalar κ. Writing this matrix equation as two scalar equations and then eliminating κ gives the stated equation.

For the second part, if the second fundamental form is a multiple of the first, the Weingarten map is a scalar multiple of the identity map, so every tangent vector is principal and every curve on the surface is a line of curvature. If $F = M = 0$ the matrices $\mathcal{F}_I$ and $\mathcal{F}_{II}$ are diagonal, hence so is the matrix $\mathcal{F}_I^{-1}\mathcal{F}_{II}$ of the Weingarten map with respect to the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$. This means that $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are principal vectors, i.e. that the parameter curves v = constant and u = constant are lines of curvature.

Conversely, if every parameter curve is a line of curvature, the stated equation must hold if $\dot{u} = 0$ and if $\dot{v} = 0$. This gives $EM = FL$ and $FN = GM$, which imply that $(EN - GL)F = EGM - EGM = 0$ and so either $EN = GL$ or $F = 0$. If $F = 0$ then $GM = 0$ so $M = 0$ and (ii) holds. If $EN = GL$ the equation in the exercise implies that every curve is a line of curvature, so every tangent vector is principal, so (i) holds.

Condition (i) implies that the two principal curvatures are equal everywhere, i.e. every point is an umbilic, so $\boldsymbol{\sigma}$ is an open subset of a plane or a sphere by Proposition 8.2.9.

From Examples 6.1.3 and 7.1.2, the first and second fundamental forms of a surface of revolution are $du^2 + f(u)^2 dv^2$ and $(\dot{f}\ddot{g} - \ddot{f}\dot{g})du^2 + f\dot{g}dv^2$, respectively. Since the terms $du dv$ are absent, the vectors $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are principal; but these are tangent to the meridians and parallels, respectively.

8.2.4 Let $\mathbf{N}_1$ be a unit normal of $\mathcal{S}$. Then, $K = 0 \iff \dot{\mathbf{N}}_1.(\mathbf{t} \times \mathbf{N}_1) = 0$. Since $\dot{\mathbf{N}}_1$ is perpendicular to $\mathbf{N}_1$ and $\dot{\mathbf{N}}_1$ is perpendicular to $\mathbf{t}$, this condition holds $\iff \dot{\mathbf{N}}_1$ is parallel to $\mathbf{t}$, i.e. $\iff \dot{\mathbf{N}}_1 = -\lambda\dot{\boldsymbol{\gamma}}$ for some scalar λ. Now use Exercise 8.2.2.

8.2.5 Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be unit normals of the two surfaces; if $\boldsymbol{\gamma}$ is a unit-speed parametrization of $\mathcal{C}$, then $\dot{\mathbf{N}}_1 = -\lambda_1\dot{\boldsymbol{\gamma}}$ for some scalar λ_1 by Exercise 8.2.2. If $\mathcal{C}$ is a line of curvature of $\mathcal{S}_2$, then $\dot{\mathbf{N}}_2 = -\lambda_2\dot{\boldsymbol{\gamma}}$ for some scalar λ_2, and then $(\mathbf{N}_1.\mathbf{N}_2)^{\cdot} = -\lambda_1\dot{\boldsymbol{\gamma}}.\mathbf{N}_2 - \lambda_2\dot{\boldsymbol{\gamma}}.\mathbf{N}_1 = 0$, so $\mathbf{N}_1.\mathbf{N}_2$ is constant along $\boldsymbol{\gamma}$, showing that the angle between $\mathcal{S}_1$ and $\mathcal{S}_2$ is constant. Conversely, if $\mathbf{N}_1.\mathbf{N}_2$ is constant, then $\mathbf{N}_1.\dot{\mathbf{N}}_2 = 0$ since $\dot{\mathbf{N}}_1.\mathbf{N}_2 = -\lambda_1\dot{\boldsymbol{\gamma}}.\mathbf{N}_2 = 0$; thus, $\dot{\mathbf{N}}_2$ is perpendicular to $\mathbf{N}_1$, and is also perpendicular to $\mathbf{N}_2$ as $\mathbf{N}_2$ is a unit vector; but $\dot{\boldsymbol{\gamma}}$ is also perpendicular to $\mathbf{N}_1$ and $\mathbf{N}_2$; hence, $\dot{\mathbf{N}}_2$ must be parallel to $\dot{\boldsymbol{\gamma}}$, so there is a scalar λ_2 (say) such that $\dot{\mathbf{N}}_2 = -\lambda_2\dot{\boldsymbol{\gamma}}$.

8.2.6 (i) Differentiate the three equations in (8.5) with respect to w, u and v, respectively; this gives

$$\boldsymbol{\sigma}_{uw}.\boldsymbol{\sigma}_v + \boldsymbol{\sigma}_u.\boldsymbol{\sigma}_{vw} = 0, \ \boldsymbol{\sigma}_{uv}.\boldsymbol{\sigma}_w + \boldsymbol{\sigma}_v.\boldsymbol{\sigma}_{uw} = 0, \ \boldsymbol{\sigma}_{vw}.\boldsymbol{\sigma}_u + \boldsymbol{\sigma}_w.\boldsymbol{\sigma}_{uv} = 0.$$

Subtracting the second equation from the sum of the other two gives $\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_{vw} = 0$, and similarly $\boldsymbol{\sigma}_v.\boldsymbol{\sigma}_{uw} = \boldsymbol{\sigma}_w.\boldsymbol{\sigma}_{uv} = 0$.
(ii) Since $\boldsymbol{\sigma}_v.\boldsymbol{\sigma}_w = 0$, it follows that the matrix $\mathcal{F}_I$ for the $u = u_0$ surface is diagonal (and similarly for the others). Let $\mathbf{N}$ be the unit normal of the $u = u_0$ surface; $\mathbf{N}$ is parallel to $\boldsymbol{\sigma}_v \times \boldsymbol{\sigma}_w$ by definition, and hence to $\boldsymbol{\sigma}_u$ since $\boldsymbol{\sigma}_u$, $\boldsymbol{\sigma}_v$ and $\boldsymbol{\sigma}_w$ are perpendicular; by (i), $\boldsymbol{\sigma}_{vw}.\boldsymbol{\sigma}_u = 0$, hence $\boldsymbol{\sigma}_{vw}.\mathbf{N} = 0$, proving that the matrix $\mathcal{F}_{II}$ for the $u = u_0$ surface is diagonal.
(iii) By part (ii), the parameter curves of each surface $u = u_0$ are lines of curvature. But the parameter curve $v = v_0$, say, on this surface is the curve of intersection of the $u = u_0$ surface with the $v = v_0$ surface.

8.2.7 On the open subset of the ellipsoid with $z \neq 0$, we can use the parametrization $\boldsymbol{\sigma}(x, y) = (x, y, z)$, where $z = \pm r\sqrt{1 - \frac{x^2}{p^2} - \frac{y^2}{q^2}}$. By Proposition 8.1.2 and the remarks following Proposition 8.2.1, the condition for an umbilic is that $\mathcal{F}_{II} = \kappa\mathcal{F}_I$ for some scalar κ. The formulas in the solution of Exercise 8.1.1 lead to the equations $z_{xx} = \lambda(1 + z_x^2)$, $z_{xy} = \lambda z_x z_y$, $z_{yy} = \lambda(1 + z_y^2)$, where $\lambda = \kappa\sqrt{1 + z_x^2 + z_y^2}$. If x and y are both non-zero, the middle equation gives $\lambda = -1/z$, and substituting into the first equation gives the contradiction $p^2 = r^2$. Hence, either $x = 0$ or $y = 0$. If $x = 0$, the equations have the four solutions

$$x = 0, \quad y = \pm q\sqrt{\frac{q^2 - p^2}{q^2 - r^2}}, \quad z = \pm r\sqrt{\frac{r^2 - p^2}{r^2 - q^2}}.$$

Similarly, one finds the following eight other candidates for umbilics:

$$x = \pm p\sqrt{\frac{p^2 - q^2}{p^2 - r^2}}, \quad y = 0, \quad z = \pm r\sqrt{\frac{r^2 - q^2}{r^2 - p^2}},$$

$$x = \pm p\sqrt{\frac{p^2 - r^2}{p^2 - q^2}}, \quad y = \pm q\sqrt{\frac{q^2 - r^2}{q^2 - p^2}}, \quad z = 0.$$

Of these 12 points, exactly 4 are real, depending on the relative sizes of p^2, q^2 and r^2.

If $p = q \neq r$, the only umbilics are the two points $(0, 0, \pm r)$. If $p = q = r$ every point of the ellipsoid (now a sphere) is an umbilic.

8.2.8 By Proposition 8.2.3, the principal curvatures are the roots of $\kappa^2 - 2H\kappa + K = 0$, i.e. $H \pm \sqrt{H^2 - K}$. If there are no umbilics, we must have $H^2 > K$, and then the principal curvatures are smooth because H and K are (Exercise 8.1.3). The second part follows from Exercises 6.1.4 and 7.1.3 and Proposition 8.2.6.

8.2.9 Setting $x = r\cos\theta$, $y = r\sin\theta$ gives $z = a\theta$ so a parametrization of the surface is $\boldsymbol{\sigma}(r, \theta) = (r\cos\theta, r\sin\theta, a\theta)$. Then $\boldsymbol{\sigma}_r = (\cos\theta, \sin\theta, 0)$, $\boldsymbol{\sigma}_\theta = (-r\sin\theta, r\cos\theta, a)$, so $E = 1$, $F = 0$, $G = r^2 + a^2$. Next, $\boldsymbol{\sigma}_r \times \boldsymbol{\sigma}_\theta = (a\sin\theta, -a\cos\theta, r)$ so $\mathbf{N} = G^{-1/2}(a\sin\theta, -a\cos\theta, r)$. From $\boldsymbol{\sigma}_{rr} = \mathbf{0}$, $\boldsymbol{\sigma}_{r\theta} = (-\sin\theta, \cos\theta, 0)$, $\boldsymbol{\sigma}_{\theta\theta} = (-r\cos\theta, -r\sin\theta, 0)$ we get $L = 0$, $M = -aG^{-1/2}$, $N = 0$. The principal curvatures are the roots of

$$\begin{vmatrix} -\kappa & -aG^{-1/2} \\ -aG^{-1/2} & -\kappa G \end{vmatrix} = 0,$$

i.e. $\kappa = \pm a/G = \frac{\pm a}{r^2 + a^2} = \frac{\pm a}{x^2 + y^2 + a^2}$.

8.2.10 In the notation of Proposition 8.2.3, $H^2 - K = \frac{1}{4}(\kappa_1 + \kappa_2)^2 - \kappa_1\kappa_2 = \frac{1}{4}(\kappa_1 - \kappa_2)^2$.

8.2.11 The condition $H^2 = K$ gives $x^2 + y^2 + z^2 = 3$. We showed in Exercise 5.1.3 that the points (x, y, z) which satisfy this condition as well as $xyz = 1$ are precisely the four points $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$ and $(1, 1, 1)$, and we found in Exercise 8.1.13 that these are the points where the Gaussian curvature attains its maximum value.

8.2.12 If $K > 0$, then since $K = \kappa_1\kappa_2$, either $\kappa_1 > 0$ and $\kappa_2 > 0$, or $\kappa_1 < 0$ and $\kappa_2 < 0$. It follows from Euler's theorem 8.2.4 that the normal curvature κ_n of any curve $\boldsymbol{\gamma}$ on the surface is > 0 in the first case and < 0 in the second (since $\sin\theta$ and $\cos\theta$ cannot be zero simultaneously). By Eq. 7.9, the curvature of $\boldsymbol{\gamma}$ is > 0.

8.2.13 Changing the sign of $\mathbf{N}$ changes the sign of the Weingarten map $\mathcal{W}$. With the notation in Proposition 8.2.1, $(-\mathcal{W})(\mathbf{t}_1) = (-\kappa_1)\mathbf{t}_1$, $(-\mathcal{W})(\mathbf{t}_2) = (-\kappa_2)\mathbf{t}_2$, which shows that $-\kappa_1$ and $-\kappa_2$ are the principal curvatures and $\mathbf{t}_1$ and $\mathbf{t}_2$ are corresponding principal vectors.

8.2.14 Let $\boldsymbol{\sigma}(u,v)$ be a surface patch, let $T(\mathbf{v}) = P\mathbf{v} + \mathbf{a}$ be an isometry of $\mathbb{R}^3$, where P is an orthogonal matrix and $\mathbf{a} \in \mathbb{R}^3$, and let $\tilde{\boldsymbol{\sigma}} = T(\boldsymbol{\sigma})$. Then, $\tilde{\boldsymbol{\sigma}}_u = P(\boldsymbol{\sigma}_u)$, etc. so in the obvious notation $\tilde{E} = P(\boldsymbol{\sigma}_u).P(\boldsymbol{\sigma}_u) = \boldsymbol{\sigma}_u.\boldsymbol{\sigma}_u$ (since P is orthogonal) so $\tilde{E} = E$. Similarly, $\tilde{F} = F, \tilde{G} = G$. By Proposition A.1.6, $\tilde{\mathbf{N}} = \varepsilon\mathbf{N}$, the sign ε being $+$ if T is direct and $-$ if T is opposite, so $\tilde{L} = \varepsilon L$, $\tilde{M} = \varepsilon M$, $\tilde{N} = \varepsilon N$. The result now follows from Proposition 8.2.6.

8.2.15 Let κ_1 and κ_2 be the principal curvatures of a surface patch of $\mathcal{S}$ at $\mathbf{p}$ and θ the oriented angle between a principal vector corresponding to κ_1 and the tangent vector of one of the m curves at $\mathbf{p}$. By Euler's Theorem 8.2.4, the sum of the normal curvatures of the m curves at $\mathbf{p}$ is

$$\sum_{r=0}^{m-1} \left(\kappa_1 \cos^2\left(\theta + \frac{r\pi}{m}\right) + \kappa_2 \sin^2\left(\theta + \frac{r\pi}{m}\right) \right).$$

Using $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, etc., the sum becomes

$$\frac{1}{2}\sum_{r=0}^{m-1}(\kappa_1 + \kappa_2) + \frac{1}{4}(\kappa_1 - \kappa_2)\sum_{r=0}^{m-1}\left(e^{2i(\theta + r\pi/m)} + e^{-2i(\theta + r\pi/m)}\right).$$

But, $\sum_{r=0}^{m-1} e^{\pm 2i(\theta + r\pi/m)} = e^{\pm 2i\theta}\frac{1 = e^{\pm 2i\pi}}{1 - e^{\pm 2i\pi/m}} = 0$, so the desired sum is simply

$$\frac{1}{2}\sum_{r=0}^{m-1}(\kappa_1 + \kappa_2) = \frac{1}{2}m(\kappa_1 + \kappa_2) = mH,$$

where H is the mean curvature of $\mathcal{S}$ at $\mathbf{p}$.

8.2.16 Using Proposition 6.2.5 and Exercise 7.1.6 (and the notation there), $E = 1 + v^2\kappa^2$, $F = G = 1$, $L = -\kappa\tau v$, $M = N = 0$ (we work in the part of the surface in which $v > 0$; the region $v < 0$ is treated similarly). By Exercise 8.2.3 the lines of curvature are given by $\kappa\tau v\dot{u}^2 + \kappa\tau v\dot{u}\dot{v} = 0$, so $\dot{u} = 0$ or $\dot{u} + \dot{v} = 0$, i.e. $u =$ constant or $u + v =$ constant; the first of these corresponds to the tangent lines to the given curve.

8.2.17 The surface is parametrized by $\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\mathbf{N}_1(u)$ in the notation of Exercise 8.2.4. Since $\boldsymbol{\gamma}$ is a line of curvature, we have $\dot{\mathbf{N}}_1 = -\kappa\dot{\boldsymbol{\gamma}}$, where κ is the principal curvature corresponding to the principal vector $\dot{\boldsymbol{\gamma}}$.
(i) If the surface is a generalized cone, the rulings all pass through some fixed point, so there is a function $v(u)$ such that $\boldsymbol{\sigma}(u, v(u))$ is independent of u. Differentiation with respect to u gives $\dot{\boldsymbol{\gamma}} + \dot{v}\mathbf{N}_1 - \kappa v\dot{\boldsymbol{\gamma}} = \mathbf{0}$. Since $\dot{\boldsymbol{\gamma}}$ is perpendicular to $\mathbf{N}_1$, this gives $1 - \kappa v = 0$ and $\dot{v} = 0$, so $\kappa = 1/v$ is a non-zero constant. Conversely, if κ is constant, $\boldsymbol{\sigma}(u, 1/\kappa)$ is constant so the surface is a generalized cone.

(ii) If the surface is a generalized cylinder, the rulings are all parallel, so $\mathbf{N}_1$ is constant along $\boldsymbol{\gamma}$. So $\dot{\mathbf{N}}_1 = \mathbf{0}$, and since $\dot{\mathbf{N}}_1 = -\kappa\dot{\boldsymbol{\gamma}}$ we have $\kappa = 0$. Conversely, if $\kappa = 0$ then $\dot{\mathbf{N}}_1 = 0$ so $\mathbf{N}_1$ is constant along $\boldsymbol{\gamma}$ and the rulings are parallel.

8.2.18 The image of $\boldsymbol{\gamma}$ under the Gauss map is the curve $\mathbf{N}$, so the condition is that $\dot{\boldsymbol{\gamma}}$ is parallel to $\dot{\mathbf{N}}$, i.e. $\dot{\mathbf{N}} = \lambda\dot{\boldsymbol{\gamma}}$ for some scalar λ. The result now follows from Exercise 8.2.2. The second part follows from the last sentence of Proposition 8.2.1.

8.2.19 If every curve on $\mathcal{S}$ is a line of curvature, every tangent vector of $\mathcal{S}$ at $\mathbf{p}$ is a principal vector. By Proposition 8.2.6, it is enough to prove that every point of $\mathcal{S}$ is an umbilic. Suppose for a contradiction that $\mathbf{p} \in \mathcal{S}$ is not an umbilic, and let $\kappa_1 \neq \kappa_2$ be the principal curvatures at $\mathbf{p}$ with corresponding non-zero principal vectors $\mathbf{t}_1$ and $\mathbf{t}_2$. If $\mathcal{W}$ is the Weingarten map of $\mathcal{S}$ at $\mathbf{p}$, we have $\mathcal{W}(\mathbf{t}_1 + \mathbf{t}_2) = \kappa_1\mathbf{t}_1 + \kappa_2\mathbf{t}_2$. Since every tangent vector is principal, there is a scalar λ such that $\mathcal{W}(\mathbf{t}_1 + \mathbf{t}_2) = \lambda(\mathbf{t}_1 + \mathbf{t}_2)$; then $(\kappa_1 - \lambda)\mathbf{t}_1 + (\kappa_2 - \lambda)\mathbf{t}_2 = \mathbf{0}$. But $\mathbf{t}_1$ and $\mathbf{t}_2$ are linearly independent, so this equation forces $\kappa_1 = \lambda = \kappa_2$, a contradiction.

8.2.20 In the notation of Exercise 7.3.22, $\tau_g = \dot{\mathbf{N}}.(\mathbf{t} \times \mathbf{N})$. By Exercise 8.2.2, if $\boldsymbol{\gamma}$ is a line of curvature $\dot{\mathbf{N}}$ is parallel to $\mathbf{t}$, so $\tau_g = 0$. Conversely, if $\tau_g = 0$ then $\mathbf{N}.(\dot{\mathbf{N}} \times \mathbf{t}) = 0$; but $\dot{\mathbf{N}}$ and $\mathbf{t}$ are both perpendicular to $\mathbf{N}$, so $\dot{\mathbf{N}} \times \mathbf{t}$ is parallel to $\mathbf{N}$; hence, $\dot{\mathbf{N}} \times \mathbf{t} = \mathbf{0}$, so $\dot{\mathbf{N}}$ is parallel to $\mathbf{t}$ and $\boldsymbol{\gamma}$ is a line of curvature by Exercise 8.2.2 again.

8.2.21 We use the usual Frenet-Serret notation for $\boldsymbol{\gamma}$ and let $\mathbf{N}$ be the unit normal of $\mathcal{S}$. We are given that $\mathbf{b}.\mathbf{N}$ is constant, so $\tau\mathbf{n}.\mathbf{N} = \mathbf{b}.\dot{\mathbf{N}}$. If κ is the principal curvature of $\mathcal{S}$ corresponding to $\boldsymbol{\gamma}$, we have $\dot{\mathbf{N}} = -\kappa\mathbf{t}$ so $\mathbf{b}.\dot{\mathbf{N}} = 0$. If $\tau \neq 0$ at some point $\boldsymbol{\gamma}(t_0)$ of $\boldsymbol{\gamma}$, say, then for some $\epsilon > 0$ we have $\tau(t) \neq 0$ if $t_0 - \epsilon < t < t_0 + \epsilon$. For values of t in this interval, $\mathbf{n}$ is perpendicular to $\mathbf{N}$, so $\mathbf{b}$ is parallel to $\mathbf{N}$, hence $\mathbf{b} = \pm\mathbf{N}$. But then $\dot{\mathbf{b}} = \pm\dot{\mathbf{N}}$ so $\tau\mathbf{n} = \pm\kappa\mathbf{t}$; this forces $\tau = \kappa = 0$, contradicting assumption. It follows that $\tau = 0$ at all points of $\boldsymbol{\gamma}$, so $\boldsymbol{\gamma}$ is a plane curve.

The converse is a special case of Exercise 8.2.5, since every curve in a plane is a line of curvature of the plane.

8.2.22 In the notation of the proof of Euler's Theorem 8.2.4, $\mathbf{t} = \cos\theta\mathbf{t}_1 + \sin\theta\mathbf{t}_2$, so $\dot{\mathbf{N}} = -\mathcal{W}(\mathbf{t}) = -\kappa_1\cos\theta\mathbf{t}_1 - \kappa_2\sin\theta\mathbf{t}_2$. Since $\mathbf{t}_2 \times \mathbf{N} = \mathbf{t}_1$ and $\mathbf{t}_1 \times \mathbf{N} = -\mathbf{t}_2$, we have $\mathbf{t} \times \mathbf{N} = -\cos\theta\mathbf{t}_2 + \sin\theta\mathbf{t}_1$ so since $\mathbf{t}_1$ and $\mathbf{t}_2$ are perpendicular unit vectors, $\tau_g = \dot{\mathbf{N}}.(\mathbf{t} \times \mathbf{N}) = -\kappa_1\sin\theta\cos\theta + \kappa_2\sin\theta\cos\theta = (\kappa_2 - \kappa_1)\sin\theta\cos\theta$.

8.2.23 By Euler's Theorem 8.2.4, for an asymptotic curve we have $\kappa_1\cos^2\theta + \kappa_2\sin^2\theta = 0$, so $\kappa_1\kappa_2 = -\kappa_2^2\tan^2\theta \leq 0$ (if θ is a multiple of $\pi/2$ Euler's Theorem gives $\kappa_1 = 0$ or $\kappa_2 = 0$, and the condition again holds). If $\mathcal{S}$ is ruled, we must have $\kappa_1\kappa_2 \leq 0$ because the rulings are asymptotic curves (Exercise 7.3.6).

If $\kappa_1\kappa_2 < 0$, Euler's theorem gives $\tan^2\theta = -\kappa_1/\kappa_2$, so there are two asymptotic

curves each of which makes an angle $\tan^{-1}\sqrt{-\kappa_1/\kappa_2}$ with a principal vector corresponding to κ_1, so the angle between them is as stated.

If $\kappa_1 = \kappa_2 = 0$ everywhere, every curve is asymptotic. If $\kappa_1 = 0$ but $\kappa_2 \neq 0$, say, Euler's Theorem gives $\sin\theta = 0$ so there is one asymptotic curve through each point, namely the line of curvature corresponding to κ_1 (it is proved in Proposition 8.4.2 that these curves are, in fact, straight lines).

8.2.24 Part (i) follows from the fact that $\langle\!\langle\ ,\ \rangle\!\rangle$ is symmetric; part (ii) follows from the fact that it is linear in its second argument (since it is actually bilinear).

8.2.25 This is an immediate consequence of Proposition 7.3.3.

8.2.26 $\langle\!\langle \mathbf{t}_1, \mathbf{t}_2 \rangle\!\rangle = \langle \mathcal{W}(\mathbf{t}_1), \mathbf{t}_2 \rangle = \kappa_1 \langle \mathbf{t}_1, \mathbf{t}_2 \rangle$. But if $\kappa_1 \neq \kappa_2$ then $\mathbf{t}_1$ is perpendicular to $\mathbf{t}_2$ (Proposition 8.2.1).

8.2.27 By Proposition 8.2.1, we can assume that $\mathbf{t}_1$ is perpendicular to $\mathbf{t}_2$ (whether or not κ_1 and κ_2 are distinct). Then $\mathbf{t} = \cos\theta\mathbf{t}_1 + \sin\theta\mathbf{t}_2$, $\tilde{\mathbf{t}} = \cos\tilde{\theta}\mathbf{t}_1 + \sin\tilde{\theta}\mathbf{t}_2$, so

$$\langle\!\langle \mathbf{t}, \tilde{\mathbf{t}} \rangle\!\rangle = \cos\theta\cos\tilde{\theta}\langle\!\langle \mathbf{t}_1, \mathbf{t}_1 \rangle\!\rangle + (\cos\theta\sin\tilde{\theta} + \cos\tilde{\theta}\sin\theta)\langle\!\langle \mathbf{t}_1, \mathbf{t}_2 \rangle\!\rangle + \sin\theta\sin\tilde{\theta}\langle\!\langle \mathbf{t}_2, \mathbf{t}_2 \rangle\!\rangle.$$

Now $\langle\!\langle \mathbf{t}_i, \mathbf{t}_j \rangle\!\rangle = \langle \mathcal{W}(\mathbf{t}_i), \mathbf{t}_j \rangle = \kappa_i \langle \mathbf{t}_i, \mathbf{t}_j \rangle = \kappa_i$ if $i = j$ and $= 0$ otherwise. Hence, $\langle\!\langle \mathbf{t}, \tilde{\mathbf{t}} \rangle\!\rangle = \kappa_1 \cos\theta\cos\tilde{\theta} + \kappa_2 \sin\theta\sin\tilde{\theta}$, from which the result follows.

8.2.28 Using the notation and result of the preceding exercise,

$$\begin{aligned}
\frac{1}{\kappa_n} + \frac{1}{\tilde{\kappa}_n} &= \frac{1}{\kappa_1\cos^2\theta + \kappa_2\sin^2\theta} + \frac{1}{\kappa_1\cos^2\tilde{\theta} + \kappa_2\sin^2\tilde{\theta}} \\
&= \frac{1 + \tan^2\theta}{\kappa_1 + \kappa_2\tan^2\theta} + \frac{1 + \tan^2\tilde{\theta}}{\kappa_1 + \kappa_2\tan^2\tilde{\theta}} \\
&= \frac{1}{\kappa_1} + \frac{1}{\kappa_2} + \frac{\left(1 - \frac{\kappa_2}{\kappa_1}\right)\tan^2\theta}{\kappa_1 + \kappa_2\tan^2\theta} + \frac{1 - \frac{\kappa_1}{\kappa_2}}{\kappa_1 + \frac{\kappa_1^2}{\kappa_2}\cot^2\theta} \\
&= \frac{1}{\kappa_1} + \frac{1}{\kappa_2}.
\end{aligned}$$

8.2.29 $\langle\!\langle \mathbf{v}, \dot{\boldsymbol{\gamma}} \rangle\!\rangle = \langle \mathbf{v}, \mathcal{W}(\dot{\boldsymbol{\gamma}}) \rangle = -\langle \mathbf{v}, \dot{\mathbf{N}} \rangle = -\mathbf{v}.\dot{\mathbf{N}} = \dot{\mathbf{v}}.\mathbf{N}$ (since $\mathbf{v}.\mathbf{N} = 0$).

8.2.30 The formulas for E^*, F^* and G^* were proved in Exercise 6.3.11, where it was also shown that

$$\boldsymbol{\sigma}_u^* = \frac{\boldsymbol{\sigma}_u}{\parallel \boldsymbol{\sigma} \parallel^2} - \frac{2(\boldsymbol{\sigma}_u.\boldsymbol{\sigma})\boldsymbol{\sigma}}{\parallel \boldsymbol{\sigma} \parallel^4}.$$

Hence,

$$\boldsymbol{\sigma}_{uu}^* = \frac{\boldsymbol{\sigma}_{uu}}{\parallel \boldsymbol{\sigma} \parallel^2} - \frac{4(\boldsymbol{\sigma}_u.\boldsymbol{\sigma})\boldsymbol{\sigma}_u}{\parallel \boldsymbol{\sigma} \parallel^4} - \frac{2(\boldsymbol{\sigma}_{uu}.\boldsymbol{\sigma} + \boldsymbol{\sigma}_u.\boldsymbol{\sigma}_u)\boldsymbol{\sigma}}{\parallel \boldsymbol{\sigma} \parallel^4} + \frac{8(\boldsymbol{\sigma}_u.\boldsymbol{\sigma})^2\boldsymbol{\sigma}}{\parallel \boldsymbol{\sigma} \parallel^6}.$$

Using the formula for $\mathbf{N}^*$ in Exercise 4.5.5, this gives

$$L^* = \left(\frac{\boldsymbol{\sigma}_{uu}}{\parallel \boldsymbol{\sigma} \parallel^2} - \frac{4(\boldsymbol{\sigma}_u.\boldsymbol{\sigma})\boldsymbol{\sigma}_u}{\parallel \boldsymbol{\sigma} \parallel^4} - \frac{2(\boldsymbol{\sigma}_{uu}.\boldsymbol{\sigma} + \boldsymbol{\sigma}_u.\boldsymbol{\sigma}_u)\boldsymbol{\sigma}}{\parallel \boldsymbol{\sigma} \parallel^4} + \frac{8(\boldsymbol{\sigma}_u.\boldsymbol{\sigma})^2\boldsymbol{\sigma}}{\parallel \boldsymbol{\sigma} \parallel^6}\right) . \left(\frac{2(\boldsymbol{\sigma}.\mathbf{N})\boldsymbol{\sigma}}{\parallel \boldsymbol{\sigma} \parallel^2} - \mathbf{N}\right),$$

which simplifies to give

$$L^* = -\frac{(\boldsymbol{\sigma}_{uu}.\mathbf{N})}{\parallel \boldsymbol{\sigma} \parallel^2} - \frac{2(\boldsymbol{\sigma}.\mathbf{N})(\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_u)}{\parallel \boldsymbol{\sigma} \parallel^4} = -\frac{L}{\parallel \boldsymbol{\sigma} \parallel^2} - \frac{2E(\boldsymbol{\sigma}.\mathbf{N})\boldsymbol{\sigma}}{\parallel \parallel^4}E.$$

Similar calculations give the stated formulas for M^* and N^*.
In the usual notation, the preceding formulas can be written

$$\mathcal{F}_I^* = \frac{1}{\parallel \mathbf{p} \parallel^4}\mathcal{F}_I, \quad \mathcal{F}_{II}^* = -\frac{1}{\parallel \mathbf{p} \parallel^2}\mathcal{F}_{II} - \frac{2(\mathbf{p}.\mathbf{N})}{\parallel \mathbf{p} \parallel^4}\mathcal{F}_I.$$

Hence, for any $\kappa \in \mathbb{R}$,

$$\mathcal{F}_{II}^* + (\parallel \mathbf{p} \parallel^2 \kappa + 2(\mathbf{p}.\mathbf{N}))\mathcal{F}_I^* = -\frac{1}{\parallel \mathbf{p} \parallel^2}(\mathcal{F}_{II} - \kappa\mathcal{F}_I).$$

Parts (i) and (ii) follow immediately from this equation, and part (iii) is an immediate consequence of (ii).

8.2.31 In the notation of the proof of Proposition 8.1.2, we have $\mathbf{N}_u = a\boldsymbol{\sigma}_u + b\boldsymbol{\sigma}_v$, $\mathbf{N}_v = c\boldsymbol{\sigma}_u + d\boldsymbol{\sigma}_v$, so

$$\begin{aligned}
\mathcal{F}_{III} &= \begin{pmatrix} \mathbf{N}_u.\mathbf{N}_u & \mathbf{N}_u.\mathbf{N}_v \\ \mathbf{N}_u.\mathbf{N}_v & \mathbf{N}_v.\mathbf{N}_v \end{pmatrix} \\
&= \begin{pmatrix} Ea^2 + 2Fab + Gb^2 & Eac + F(ad+bc) + Gbd \\ Eac + F(ad+bc) + Gbd & Ec^2 + 2Fcd + Gd^2 \end{pmatrix} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = (-\mathcal{F}_I^{-1}\mathcal{F}_{II})^t\mathcal{F}_I(-\mathcal{F}_I^{-1}\mathcal{F}_{II}) \\
&= \mathcal{F}_{II}\mathcal{F}_I^{-1}\mathcal{F}_I\mathcal{F}_I^{-1}\mathcal{F}_{II} = \mathcal{F}_{II}\mathcal{F}_I^{-1}\mathcal{F}_{II}.
\end{aligned}$$

8.2.32 We can assume that the profile curve of $\boldsymbol{\sigma}$ is unit-speed. From Examples 6.1.3 and 7.1.2, the principal curvatures are $-\ddot{f}\dot{g} + \dot{f}\ddot{g} = -\ddot{f}/\dot{g}$ (if $\dot{g} \neq 0$) and $\dot{g}/f$.
(i) If $\dot{g}(u_0) = 0$ the first principal curvature must be non-zero when $u = u_0$, so $\ddot{g}(u_0) \neq 0$. Since $\ddot{g}$ is a continuous function of u, there is an $\epsilon > 0$ such that $\ddot{g}(u) \neq 0$ if $0 < |u - u_0| < \epsilon$. Then, $\dot{g}(u) \neq 0$ if $0 < |u - u_0| < \epsilon$ by the intermediate value theorem.
(ii) Assume that $\dot{g}$ is never zero. Then the first principal curvature must be zero everywhere, so $\ddot{f} = 0$, i.e. $f(u) = au + b$ for some constants a, b. Then,

$\dot{g}^2 = 1 - \dot{f}^2 = 1 - a^2$ so (up to a sign) $g(u) = \sqrt{1-a^2}u + c$ for some constant c. Since $\dot{g} \neq 0$, $a^2 < 1$. If $a = 0$ we have a circular cylinder; if $a \neq 0$ then making the reparametrization $\tilde{u} = au + b$ and a suitable translation parallel to the z-axis gives the circular cone $\tilde{\boldsymbol{\sigma}}(\tilde{u}, v) = \tilde{u}(\cos v, \sin v, \sqrt{1-a^2})$.

8.2.33 The first and second fundamental forms are

$$(1+z_x^2)dx^2 + 2z_x z_y dxdy + (1+z_y^2)dy^2, \quad \frac{z_{xx}dx^2 + 2z_{xy}dxdy + z_{yy}dy^2}{\sqrt{1+z_x^2+z_y^2}}.$$

At an umbilic we have $\mathcal{F}_{II} = \kappa\mathcal{F}_I$, where κ is the principal curvature. This is equivalent to

$$z_{xx} = \lambda(1+z_x^2), \quad z_{xy} = \lambda z_x z_y, \quad z_{yy} = \lambda(1+z_y^2),$$

where $\lambda = \kappa\sqrt{1+z_x^2+z_y^2}$.

8.2.34 By Exercises 6.1.2 and 7.1.4, applying an isometry of $\mathbb{R}^3$ does not change the first fundamental form, and either leaves the second fundamental form unchanged (if the isometry is direct) or changes its sign (if the isometry is opposite). It follows that the principal curvatures stay the same (if the isometry is direct) or both change sign (if the isometry is opposite). All the assertions in the exercise follow immediately from this.

8.2.35 (i) By applying an isometry of $\mathbb{R}^3$, we can assume (Theorem 5.2.2) that the hyperboloid is of the form

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = \frac{z^2}{r^2} - 1,$$

where p, q, r are positive constants, and by the preceding exercise it suffices to consider this case. Differentiating implicitly we find

$$z_x = \frac{r^2 x}{p^2 z}, \quad z_y = \frac{r^2 y}{q^2 z},$$
$$z_{xx} = \frac{r^2}{p^2 z}\left(1 - \frac{r^2 x^2}{p^2 z^2}\right), \quad z_{xy} = -\frac{r^4 xy}{p^2 q^2 z^3}, \quad z_{yy} = \frac{r^2}{q^2 z}\left(1 - \frac{r^2 y^2}{q^2 z^2}\right).$$

By Exercise 8.2.33 the conditions for an umbilic are

$$\frac{r^2}{p^2 z}\left(1 - \frac{r^2 x^2}{p^2 z^2}\right) = \lambda\left(1 + \frac{r^4 x^2}{p^4 z^2}\right),$$
$$-\frac{r^4 xy}{p^2 q^2 z^3} = \frac{\lambda r^4 xy}{p^2 q^2 z^2},$$
$$\frac{r^2}{q^2 z}\left(1 - \frac{r^2 y^2}{q^2 z^2}\right) = \lambda\left(1 + \frac{r^4 y^2}{q^4 z^2}\right).$$

The second equation gives $x = 0$, $y = 0$ or $\lambda = -1/z$; but substituting $\lambda = -1/z$ in the first equation leads to $r^2/p^2 = -1$, which is absurd; so $x = 0$ or $y = 0$.

If $x = 0$, the first equation gives $\lambda = r^2/p^2 z$ and then the last equation gives

$$p^2\left(1 - \frac{r^2y^2}{q^2z^2}\right) = q^2\left(1 + \frac{r^4y^2}{q^4z^2}\right).$$

Using $y^2/q^2 = z^2/r^2 - 1$, this leads to $y = \pm q\sqrt{\frac{p^2-q^2}{q^2+r^2}}$, $z = \pm r\sqrt{\frac{p^2+r^2}{q^2+r^2}}$ (with any combination of signs). Similarly, $y = 0$ leads to $x = \pm p\sqrt{\frac{q^2-p^2}{p^2+r^2}}$, $z = \pm r\sqrt{\frac{q^2+r^2}{p^2+r^2}}$. If $p^2 \neq q^2$, this gives four distinct (real) points of the surface; if $p^2 = q^2$ we get the two points $(0, 0, \pm r)$.

(ii) We can take the surface to be

$$z = \frac{x^2}{p^2} + \frac{y^2}{q^2}.$$

Proceeding as in (i), we find that the conditions for an umbilic are

$$\frac{2}{p^2} = \lambda\left(1 + \frac{4x^2}{p^4}\right), \quad 0 = \frac{4\lambda xy}{p^2q^2}, \quad \frac{2}{q^2} = \lambda\left(1 + \frac{4y^2}{q^4}\right).$$

The second equation gives $x = 0$, $y = 0$ or $\lambda = 0$, and $\lambda = 0$ is impossible by the other two equations. If $x = 0$ we get $y = \pm\frac{1}{2}q\sqrt{p^2 - q^2}$, $z = \pm\frac{1}{4}(p^2 - q^2)$. If $y = 0$ we get $x = \pm\frac{1}{2}p\sqrt{q^2 - p^2}$, $z = \pm\frac{1}{4}(q^2 - p^2)$. If $p^2 \neq q^2$ this gives two (real) points of the surface; if $p^2 = q^2$ we get the origin as the only umbilic.

(iii) The calculation is simplified by noting that these surfaces are ruled, so that, if κ_1, κ_2 are the principal curvatures, $\kappa_1\kappa_2 \leq 0$ everywhere (Exercise 8.2.23). Hence, if $\kappa_1 = \kappa_2$ we must have $\kappa_1 = \kappa_2 = 0$, and hence $\mathcal{F}_{II} = 0$, so we are looking for the points at which the second fundamental form is zero.

For the hyperboloid of one sheet

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = \frac{z^2}{r^2} + 1,$$

we can use x, y as parameters on the open subset of the surface where $z \neq 0$. The formulas for z_{xx}, z_{xy} and z_{yy} are the same as in part (i); the condition $z_{xy} = 0$ forces $x = 0$ or $y = 0$, but neither of these is consistent with the conditions $z_{xx} = z_{yy} = 0$. Hence, there are no umbilics for which $z \neq 0$.

To deal with the 'waist' $z = 0$ of the hyperboloid, we can use x, z as parameters if $x \neq 0$ and y, z if $y \neq 0$. In the first case, for example, the condition for an

umbilic is $y_{xx} = y_{xz} = y_{zz} = 0$; proceeding as above, we find that there are no umbilics. Similarly in the last case.

Finally, for the hyperbolic paraboloid

$$z = \frac{x^2}{p^2} - \frac{y^2}{q^2},$$

$z_{xx} = 2/p^2$ is never zero, so there are no umbilics.

8.2.36 We have to show that the umbilics on the surface $xyz = 1$ are the four points $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$ and $(-1,-1,1)$. Using x, y as parameters, we find as in the preceding exercise that the conditions for an umbilic are

$$\begin{aligned} \frac{2}{x^3y} &= \lambda\left(1 + \frac{1}{x^4y^2}\right), \\ \frac{1}{x^2y^2} &= \frac{\lambda}{x^3y^3}, \\ \frac{2}{xy^3} &= \lambda\left(1 + \frac{1}{x^2y^4}\right), \end{aligned}$$

for some λ. The second equation gives $\lambda = xy$ and then the other equations give $x^4y^2 = x^2y^4 = 1$; these equations force $x^2 = y^2 = 1$, so $x = \pm 1$, $y = \pm 1$ (with any combination of signs). Then $z = 1/xy = \pm 1$ and we obtain the four points required.

8.2.37 On the open subset of $\mathcal{S}$ on which $z \neq 0$ we can use x, y as parameters. Differentiating implicitly we find that

$$z_{xx} = \frac{(y^2+z^2)(2x^2z^2 - y^2z^2 - x^2y^2)}{z^3(x^2+y^2)^2},$$

so $z_{xx} = 0$ when $x^2 = y^2 = z^2 = 1$, and in particular at $\mathbf{p}$. Similarly, $z_{xy} = z_{yy} = 0$ at $\mathbf{p}$. It follows that the second fundamental form of $\mathcal{S}$ is zero at $\mathbf{p}$ (Exercise 8.2.33), i.e., $\mathbf{p}$ is a planar point. The argument actually proves that the eight points $(\pm 1, \pm 1, \pm 1)$ are all planar, but this can also be deduced from Exercise 8.2.34 and the fact that $\mathcal{S}$ is preserved by the isometries $(x, y, z) \mapsto (\pm x, \pm y, \pm z)$ of $\mathbb{R}^3$ (with any combination of signs).

(i) $\mathcal{S}$ is a level surface of $f(x, y, z) = x^2y^2 + y^2z^2 + z^2x^2$, so

$$\boldsymbol{\nabla} f = (2x(y^2+z^2), 2y(x^2+z^2), 2z(x^2+y^2))$$

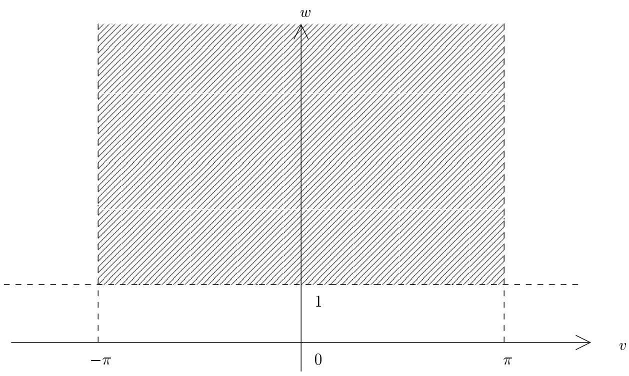
is normal to $\mathcal{S}$ at (x, y, z). Taking $x = y = 1$ we find that $(4,4,4) = 4\mathbf{n}$ is normal to $\mathcal{S}$ at $\mathbf{p}$, and hence so is $\mathbf{n}$. Since $\mathbf{t}_1$ and $\mathbf{t}_2$ are perpendicular to $\mathbf{n}$, they are tangent to $\mathcal{S}$ at $\mathbf{p}$.

(ii) Since the vectors $\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}$ are linearly independent, scalars X, Y, Z as in the statement of the exercise exist, and are small if (x, y, z) is near $\mathbf{p}$. When we substitute $x = 1 + X + Z$, $y = 1 - X + Y + Z$, $z = 1 - Y + Z$ in the equation of $\mathcal{S}$, we know that all quadratic and lower order terms will cancel, and the quartic terms can be discarded as they will be small compared to the cubic terms. Retaining only the cubic terms, we find after some algebra the equation $6Z - 3X^2Y + 3XY^2 = 0$, hence the stated equation.
(iii) Making the linear transformation $X = -\tilde{X} + \sqrt{3}\tilde{Y}$, $Y = \tilde{X} + \sqrt{3}\tilde{Y}$, $\tilde{Z} = Z$, we get $\tilde{Z} = \tilde{X}(\tilde{X}^2 - 3\tilde{Y}^2)$, which is a monkey saddle.

8.3.1 (i) Setting $\tilde{u} = v, \tilde{v} = w = e^{-u}$, we have $u = -\ln \tilde{v}, v = \tilde{u}$ so, in the notation of Exercise 6.1.4, $J = \begin{pmatrix} 0 & -\frac{1}{\tilde{v}} \\ 1 & 0 \end{pmatrix}$. Since J is invertible, $(u, v) \mapsto (v, w)$ is a reparametrization map. The first fundamental form in terms of v, w is given by $\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = J^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} J = \begin{pmatrix} 0 & 1 \\ -\frac{1}{\tilde{v}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & f(u)^2 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{\tilde{v}} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{w^2} & 0 \\ 0 & \frac{1}{w^2} \end{pmatrix}$, so the first fundamental form is $(dv^2 + dw^2)/w^2$.
(ii) We find that the matrix

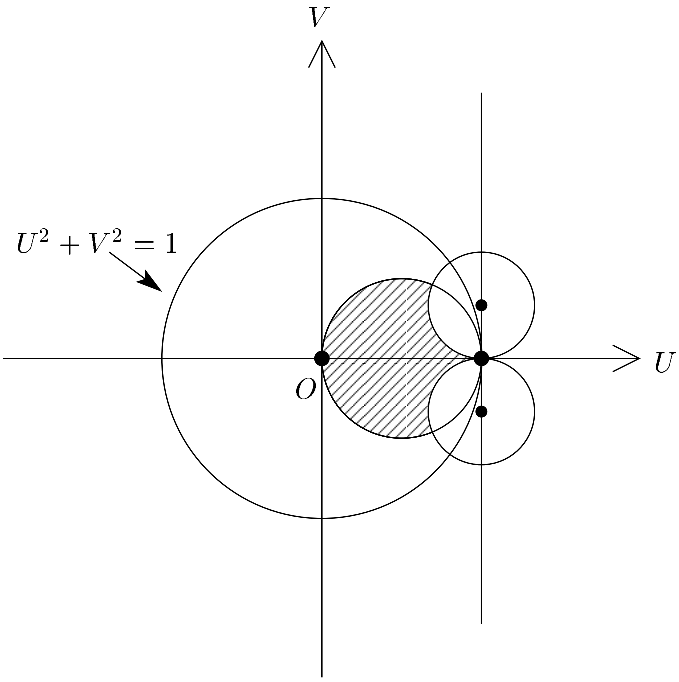
$$\tilde{J} = \begin{pmatrix} \frac{\partial v}{\partial V} & \frac{\partial v}{\partial W} \\ \frac{\partial w}{\partial V} & \frac{\partial w}{\partial W} \end{pmatrix} = \begin{pmatrix} v(w+1) & \frac{1}{2}(v^2 - (w+1)^2) \\ -\frac{1}{2}(v^2 - (w+1)^2) & v(w+1) \end{pmatrix},$$

so the first fundamental form in terms of V and W is $\tilde{J}^t \begin{pmatrix} \frac{1}{w^2} & 0 \\ 0 & \frac{1}{w^2} \end{pmatrix} \tilde{J} = \frac{(v^2+(w+1)^2)^2}{4w^2} I = \frac{4}{(1-U^2-V^2)^2} I$, after some tedious algebra.
In (i), $u < 0$ and $-\pi < v < \pi$ corresponds to $-\pi < v < \pi$ and $w > 1$, a semi-infinite rectangle in the upper half of the vw-plane.



To find the corresponding region in (ii), it is convenient to introduce the complex numbers $z = v + iw$, $Z = U + iV$. Then, the equations in (ii) are equivalent to

$Z = \frac{z-i}{z+i}$, $z = \frac{Z+1}{i(Z-1)}$. The line $v = \pi$ in the vw-plane corresponds to $z + \bar{z} = 2\pi$ (the bar denoting complex conjugate), i.e. $\frac{Z+1}{i(Z-1)} - \frac{\bar{Z}+1}{i(\bar{Z}-1)} = 2\pi$, which simplifies to $|Z - (1 - \frac{i}{\pi})|^2 = \frac{1}{\pi^2}$; so $v = \pi$ corresponds to the circle in the UV-plane with centre $1 - \frac{i}{\pi}$ and radius $\frac{1}{\pi}$. Similarly, $v = -\pi$ corresponds to the circle with centre $1 + \frac{i}{\pi}$ and radius $\frac{1}{\pi}$. Finally, $w = 1$ corresponds to $z - \bar{z} = 2i$, i.e. $\frac{Z+1}{i(Z-1)} + \frac{\bar{Z}+1}{i(\bar{Z}-1)} = 2i$. This simplifies to $|Z - \frac{1}{2}|^2 = \frac{1}{4}$; so $w = 1$ corresponds to the circle with centre $1/2$ and radius $1/2$ in the UV-plane. The required region in the UV-plane is that bounded by these three circles:



For (iii) we follow the hint and make use of polar coordinates on the disc, $V = r\cos\theta$, $W = r\sin\theta$, $\bar{V} = \bar{r}\cos\bar{\theta}$, $\bar{W} = \bar{r}\sin\bar{\theta}$. We find that $\bar{r} = \frac{2r}{r^2+1}$, $\bar{\theta} = \theta$. Suppose that the first fundamental form in terms of these parameters is $Ed\bar{r}^2 + 2Fd\bar{r}d\bar{\theta} + Gd\bar{\theta}^2$. Since $\frac{d\bar{r}}{dr} = \frac{2(1-r^2)}{(1+r^2)^2}$, the first fundamental form is $\frac{4(1-r^2)^2}{(1+r^2)^4}Edr^2 + \frac{4(1-r^2)}{(1+r^2)^2}Fdrd\theta + Gd\theta^2$. Equating this to $\frac{4(dV^2+dW^2)}{(1-V^2-W^2)^2} = \frac{4(dr^2+r^2d\theta^2)}{(1-r^2)^2}$, we get $E = \frac{(1+r^2)^4}{(1-r^2)^4} = \frac{1}{(1-\bar{r}^2)^2}$, $F = 0$, $G = \frac{4r^2}{(1-r^2)^2} = \frac{\bar{r}^2}{1-\bar{r}^2}$. Converting back to the parameters $(\bar{V}, \bar{W})$, we have $\bar{r}d\bar{r} = \bar{V}d\bar{V} + \bar{W}d\bar{W}, \bar{r}^2d\bar{\theta} =$

$\bar{V}d\bar{W} - \bar{W}d\bar{V}$, so the first fundamental form becomes

$$\frac{(\bar{V}d\bar{V} + \bar{W}d\bar{W})^2 + (1-\bar{r}^2)(\bar{V}d\bar{W} - \bar{W}d\bar{V})^2}{\bar{r}^2(1-\bar{r}^2)^2}$$
$$= \frac{(\bar{V}^2 + (1-\bar{r}^2)\bar{W}^2)d\bar{V}^2 + 2\bar{r}^2\bar{V}\bar{W}d\bar{V}d\bar{W} + (\bar{W}^2 + (1-\bar{r}^2)\bar{V}^2)d\bar{W}^2}{\bar{r}^2(1-\bar{r}^2)^2}$$
$$= \frac{(1-\bar{W}^2)d\bar{V}^2 + 2\bar{V}\bar{W}d\bar{V}d\bar{W} + (1-\bar{V}^2)d\bar{W}^2}{(1-\bar{V}^2-\bar{W}^2)^2}.$$

8.3.2 The parametrisation is $\boldsymbol{\sigma}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$, $f(u) = e^u$, $g(u) = \sqrt{1-e^{2u}} - \cosh^{-1}(e^{-u})$, $-\infty < u < 0$.
(i) A parallel $u =$ constant is a circle of radius $f(u) = e^u$, so has length $2\pi e^u$.
(ii) From Example 7.1.2, $E = 1, F = 0, G = f(u)^2$, so $d\mathcal{A}_{\boldsymbol{\sigma}} = f(u)dudv$ and the area is $\int_0^{2\pi}\int_{-\infty}^0 e^u dudv = 2\pi$.
(iii) From Examples 7.1.2 and 8.1.4, the principal curvatures are $\kappa_1 = \dot{f}\ddot{g} - \ddot{f}\dot{g} = -\ddot{f}/\dot{g} = -(e^{-2u}-1)^{-1/2}$, $\kappa_2 = f\dot{g}/f^2 = \dot{g}/f = (e^{-2u}-1)^{1/2}$.
(iv) $\kappa_1 < 0$, $\kappa_2 > 0$.

8.3.3 Let $\boldsymbol{\gamma}(u) = (f(u), 0, g(u))$ and denote d/du by a dot; by Example 8.1.4, $\ddot{f} + Kf = 0$. If $K < 0$, the general solution is $f = ae^{-\sqrt{-K}u} + be^{\sqrt{-K}u}$ where a, b are constants; the condition $f(\pi/2) = f(-\pi/2) = 0$ forces $a = b = 0$, so $\boldsymbol{\gamma}$ coincides with the z-axis, contradicting the assumptions. If $K = 0$, $f = a + bu$ and again $a = b = 0$ is forced. So we must have $K > 0$ and $f = a\cos\sqrt{K}u + b\sin\sqrt{K}u$. This time, $f(\pi/2) = f(-\pi/2) = 0$ and a, b not both zero implies that the determinant

$$\begin{vmatrix} \cos\sqrt{K}\pi/2 & \sin\sqrt{K}\pi/2 \\ \cos\sqrt{K}\pi/2 & -\sin\sqrt{K}\pi/2 \end{vmatrix} = 0.$$

This gives $\sin\sqrt{K}\pi = 0$, so $K = n^2$ for some integer $n \neq 0$. If $n = 2k$ is even, $f = b\sin 2ku$, but then $f(0) = 0$, contradicting the assumptions. If $n = 2k+1$ is odd, $f = a\cos(2k+1)u$ and $f(\pi/2(2k+1)) = 0$, which contradicts the assumptions unless $k = 0$ or -1, i.e. unless $K = (2k+1)^2 = 1$. Thus, $f = a\cos u$, $\dot{g} = \sqrt{1-\dot{f}^2} = \sqrt{1-a^2\sin^2 u}$. Now, $\dot{\boldsymbol{\gamma}} = (\dot{f}, 0, \dot{g})$ is perpendicular to the z-axis $\iff \dot{g} = 0$. So the assumptions give $\sqrt{1-a^2} = 0$, i.e. $a = \pm 1$. Then, $\boldsymbol{\gamma}(u) = (\pm\cos u, 0, \pm\sin u)$ (up to a translation along the z-axis) and $\mathcal{S}$ is the unit sphere.

8.4.1 Let $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ be a patch of $\mathcal{S}$ containing $\mathbf{p} = \tilde{\boldsymbol{\sigma}}(\tilde{u}_0, \tilde{v}_0)$. The Gaussian curvature K of $\mathcal{S}$ is < 0 at $\mathbf{p}$; since K is a smooth function of $(\tilde{u}, \tilde{v})$ (Exercise 8.1.3), $K(\tilde{u}, \tilde{v}) < 0$ for $(\tilde{u}, \tilde{v})$ in some open set $\tilde{U}$ containing $(\tilde{u}_0, \tilde{v}_0)$; then every point of $\tilde{\boldsymbol{\sigma}}(\tilde{U})$ is hyperbolic. Let κ_1, κ_2 be the principal curvatures of $\tilde{\boldsymbol{\sigma}}$, let $0 < \theta < \pi/2$

be such that $\tan\theta = \sqrt{-\kappa_1/\kappa_2}$, and let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the unit tangent vectors of $\tilde{\boldsymbol{\sigma}}$ making angles θ and $-\theta$, respectively, with the principal vector corresponding to κ_1 (see Theorem 8.2.4). Applying Proposition 8.4.3 gives the result. For the last part, put $\dot{v} = 0$ in the formula for κ_n in Proposition 7.3.5: this shows that $L = 0$ if the parameter curves $v =$ constant are asymptotic. Similarly $N = 0$ if the parameter curves $u =$ constant are asymptotic.

8.4.2 The first and second fundamental forms are $(2+v^2)du^2 + 2uvdudv + (2+u^2)dv^2$ and $-4dudv/\sqrt{4+2u^2+2v^2}$. By Exercise 8.2.3, the lines of curvature are given by

$$-(2+v^2)\dot{u}^2 + (2+u^2)\dot{v}^2 = 0,$$

so

$$\int \frac{dv}{2+v^2} = \pm \int \frac{du}{2+u^2},$$

and hence

$$\sinh^{-1}\frac{v}{\sqrt{2}} \pm \sinh^{-1}\frac{u}{\sqrt{2}} = \text{constant}.$$

We can take $s = \sinh^{-1}\frac{v}{\sqrt{2}} + \sinh^{-1}\frac{u}{\sqrt{2}}$, $t = \sinh^{-1}\frac{v}{\sqrt{2}} - \sinh^{-1}\frac{u}{\sqrt{2}}$, so $u = \sqrt{2}\sinh\frac{1}{2}(s-t)$, $v = \sqrt{2}\sinh\frac{1}{2}(s+t)$.

8.4.3 Assume that $\boldsymbol{\gamma}$ is unit-speed. Then, the ruled surface is $\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\delta}(u)$, where $\boldsymbol{\delta} = \mathbf{t}\times\mathbf{N}$, $\mathbf{t} = \dot{\boldsymbol{\gamma}}$ (a dot denoting d/du), and $\mathbf{N}$ is the unit normal of (a patch of) $\mathcal{S}$. By Example 8.1.5, the flatness condition is $\dot{\boldsymbol{\delta}}.\mathbf{N} = 0$, i.e., $(\mathbf{t}\times\dot{\mathbf{N}}).\mathbf{N} = 0$. Now, $\dot{\mathbf{N}}$ is tangent to $\mathcal{S}$, so $\mathbf{t}\times\dot{\mathbf{N}}$ is parallel to $\mathbf{N}$. Hence, $\tilde{\mathcal{S}}$ is flat if and only if $\mathbf{t}\times\dot{\mathbf{N}} = \mathbf{0}$, i.e. if and only if $\dot{\mathbf{N}}$ is parallel to $\dot{\boldsymbol{\gamma}}$. By Exercise 8.2.2, this is precisely the condition for $\boldsymbol{\gamma}$ to be a line of curvature of $\mathcal{S}$.

8.5.1 By Corollary 8.1.3 and the fact that $\boldsymbol{\sigma}$ is conformal, the mean curvature of $\boldsymbol{\sigma}$ is $H = \frac{L+N}{2E}$, so

$$(*) \qquad \boldsymbol{\sigma} \text{ is minimal } \iff L+N=0 \iff (\boldsymbol{\sigma}_{uu}+\boldsymbol{\sigma}_{vv}).\mathbf{N} = 0.$$

Obviously, then, $\boldsymbol{\sigma}$ is minimal if $\Delta\boldsymbol{\sigma} = \boldsymbol{\sigma}_{uu}+\boldsymbol{\sigma}_{vv} = \mathbf{0}$. For the converse, we have to show that $\Delta\boldsymbol{\sigma} = \mathbf{0}$ if (*) holds. It is enough to prove that $\Delta\boldsymbol{\sigma}.\boldsymbol{\sigma}_u = \Delta\boldsymbol{\sigma}.\boldsymbol{\sigma}_v = 0$, since $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v, \mathbf{N}\}$ is a basis of $\mathbb{R}^3$. We compute $\Delta\boldsymbol{\sigma}.\boldsymbol{\sigma}_u = \boldsymbol{\sigma}_{uu}.\boldsymbol{\sigma}_u + \boldsymbol{\sigma}_{vv}.\boldsymbol{\sigma}_u = \frac{1}{2}(\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_u)_u + (\boldsymbol{\sigma}_v.\boldsymbol{\sigma}_u)_v - (\boldsymbol{\sigma}_v.\boldsymbol{\sigma}_{uv}) = \frac{1}{2}(\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_u - \boldsymbol{\sigma}_v.\boldsymbol{\sigma}_v)_u + (\boldsymbol{\sigma}_v.\boldsymbol{\sigma}_u)_v$. But, since $\boldsymbol{\sigma}$ is conformal, $\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_u = \boldsymbol{\sigma}_v.\boldsymbol{\sigma}_v$ and $\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_v = 0$. Hence, $\Delta\boldsymbol{\sigma}.\boldsymbol{\sigma}_u = 0$. Similarly, $\Delta\boldsymbol{\sigma}.\boldsymbol{\sigma}_v = 0$.

The first fundamental form of the given surface patch is

$$(1+u^2+v^2)^2(du^2+dv^2),$$

so it is conformal, and $\boldsymbol{\sigma}_{uu}+\boldsymbol{\sigma}_{vv} = (-2u, 2v, 2) + (2u, -2v, -2) = \mathbf{0}$.

8.5.2 Using the formula in Exercise 8.1.1 with $f(x,y) = \ln\cos y - \ln\cos x$ gives

$$H = \frac{\sec^2 x(1+\tan^2 y) - \sec^2 y(1+\tan^2 x)}{2(1+\tan^2 x + \tan^2 y)^{3/2}} = 0.$$

8.5.3 $\mathbf{\Sigma}_u = \boldsymbol{\sigma}_u + w\mathbf{N}_u$, $\mathbf{\Sigma}_v = \boldsymbol{\sigma}_v + w\mathbf{N}_v$, $\mathbf{\Sigma}_w = \mathbf{N}$. $\mathbf{\Sigma}_u.\mathbf{\Sigma}_w = 0$ since $\boldsymbol{\sigma}_u.\mathbf{N} = \mathbf{N}_u.\mathbf{N} = 0$, and similarly $\mathbf{\Sigma}_v.\mathbf{\Sigma}_w = 0$. Finally,

$$\begin{aligned}\mathbf{\Sigma}_u.\mathbf{\Sigma}_v &= \boldsymbol{\sigma}_u.\boldsymbol{\sigma}_v + w(\boldsymbol{\sigma}_u.\mathbf{N}_v + \boldsymbol{\sigma}_v.\mathbf{N}_u) + w^2\mathbf{N}_u.\mathbf{N}_v \\ &= F - 2wM + w^2\mathbf{N}_u.\mathbf{N}_v = w^2\mathbf{N}_u.\mathbf{N}_v.\end{aligned}$$

By the proof of Proposition 8.1.2, $\mathbf{N}_u = -\frac{L}{E}\boldsymbol{\sigma}_u$, $\mathbf{N}_v = -\frac{N}{G}\boldsymbol{\sigma}_v$, so $\mathbf{N}_u.\mathbf{N}_v = \frac{LN}{EG}F = 0$. Every surface $u = u_0$ (a constant) is ruled as it is the union of the straight lines given by $v =$ constant; by Exercise 8.2.4, this surface is flat provided the curve $\boldsymbol{\gamma}(v) = \boldsymbol{\sigma}(u_0, v)$ is a line of curvature of $\mathcal{S}$, i.e. if $\boldsymbol{\sigma}_v$ is a principal vector; but this is true since the matrices $\mathcal{F}_I$ and $\mathcal{F}_{II}$ are diagonal. Similarly for the surfaces $v =$ constant.

8.5.4 We take the ruled surface to be $\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\delta}(u)$, where $\boldsymbol{\gamma}$ is unit-speed and $\boldsymbol{\delta}$ is a unit vector. Denoting d/du by a dot and $\dot{\boldsymbol{\gamma}}$ by $\mathbf{t}$, the coefficients of the first and second fundamental forms are $E = \parallel \mathbf{t} + v\dot{\boldsymbol{\delta}} \parallel^2$, $F = \mathbf{t}.\boldsymbol{\delta}$, $G = 1$, $L = (\dot{\mathbf{t}} + v\ddot{\boldsymbol{\delta}}).\mathbf{N}$, $M = \dot{\boldsymbol{\delta}}.\mathbf{N}$, $N = 0$, where $\mathbf{N} = \frac{(\mathbf{t}+v\dot{\boldsymbol{\delta}})\times\boldsymbol{\delta}}{\|(\mathbf{t}+v\dot{\boldsymbol{\delta}})\times\boldsymbol{\delta}\|}$. Then, the mean curvature

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{(\dot{\mathbf{t}} - 2(\mathbf{t}.\boldsymbol{\delta})\dot{\boldsymbol{\delta}} + v\ddot{\boldsymbol{\delta}}).\mathbf{N}}{2(EG - F^2)}.$$

If $\dot{\boldsymbol{\delta}}(u_0) \neq \mathbf{0}$ for some value of u_0, $H(u_0, v) \to 0$ as $v \to \pm\infty$, since the numerator in the expression for H is independent of v while the denominator $\to \infty$ as $v \to \pm\infty$. As H is constant, $H = 0$ everywhere, contradicting the assumption. Hence, $\dot{\boldsymbol{\delta}} = \mathbf{0}$ everywhere, $\boldsymbol{\delta}$ is constant, and the surface is a generalized cylinder.

We can now assume that $\boldsymbol{\gamma}$ lies in a plane perpendicular to $\boldsymbol{\delta}$ (Exercise 5.3.3). Then, $\mathbf{t} \times \boldsymbol{\delta}$ is a unit vector and the formula for the mean curvature simplifies to

$$H = \frac{\dot{\mathbf{t}}.(\mathbf{t} \times \boldsymbol{\delta})}{2 \parallel \mathbf{t} \times \boldsymbol{\delta} \parallel^2}.$$

Now $\dot{\mathbf{t}} = \kappa\mathbf{n}$ where κ is the curvature of $\boldsymbol{\gamma}$ and $\mathbf{n}$ is its principal normal. Also, $\mathbf{t}\times\boldsymbol{\delta}$ is perpendicular to $\mathbf{t}$ and parallel to the plane containing $\boldsymbol{\gamma}$; hence, $\mathbf{t} \times \boldsymbol{\delta} = \pm\mathbf{n}$. It follows that $H = \pm\kappa/2$, so if H is a non-zero constant then κ is constant and the plane curve $\boldsymbol{\gamma}$ is a circle. Thus, the surface is a circular cylinder.

8.5.5 We use the notation and results of Proposition 8.5.2. If $\boldsymbol{\gamma}$ is a curve on $\boldsymbol{\sigma}$, the 'corresponding' curve $\boldsymbol{\gamma}^\lambda$ on the parallel surface $\boldsymbol{\sigma}^\lambda$ is $\boldsymbol{\gamma}^\lambda = \boldsymbol{\gamma} + \lambda\mathbf{N}$. Now, $\boldsymbol{\gamma}^\lambda$

is a line of curvature on $\boldsymbol{\sigma}^\lambda \iff \dot{\mathbf{N}}^\lambda = -\kappa\dot{\boldsymbol{\gamma}}^\lambda$ for some scalar κ (Exercise 8.2.2) $\iff \epsilon\dot{\mathbf{N}} = -\kappa(\dot{\boldsymbol{\gamma}} + \lambda\dot{\mathbf{N}})$. But this last equation holds if and only if $\dot{\mathbf{N}}$ is a scalar multiple of $\dot{\boldsymbol{\gamma}}$, i.e. if and only if $\boldsymbol{\gamma}$ is a line of curvature on $\boldsymbol{\sigma}$. In this case, $\dot{\boldsymbol{\gamma}}^\lambda = \dot{\boldsymbol{\gamma}} + \lambda\dot{\mathbf{N}} = (1-\kappa\lambda)\dot{\boldsymbol{\gamma}}$ is parallel to $\dot{\boldsymbol{\gamma}}$.

8.5.6 Let $\boldsymbol{\sigma}(u,v)$ be a patch of $\mathcal{S}$. We are given that $\tilde{\boldsymbol{\sigma}}(u,v) = \boldsymbol{\sigma}(u,v)+\lambda(u,v)\mathbf{N}(u,v)$ is a patch of $\tilde{\mathcal{S}}$, for some function λ, where $\mathbf{N}(u,v)$ is the unit normal of $\boldsymbol{\sigma}$ at $\boldsymbol{\sigma}(u,v)$, and that $\mathbf{N}(u,v)$ is also normal to $\tilde{\mathcal{S}}$ at $\tilde{\boldsymbol{\sigma}}(u,v)$. Now $\tilde{\boldsymbol{\sigma}}_u = \boldsymbol{\sigma}_u + \lambda\mathbf{N}_u + \lambda_u\mathbf{N}$; since $\boldsymbol{\sigma}_u$, $\tilde{\boldsymbol{\sigma}}_u$ and $\mathbf{N}_u$ are all perpendicular to $\mathbf{N}$, this implies that $\lambda_u = 0$; similarly $\lambda_v = 0$. Hence, λ is constant and $\tilde{\mathcal{S}} = \mathcal{S}^\lambda$ is a parallel surface of $\mathcal{S}$.

Chapter 9

9.1.1 By Exercise 4.1.3, there are two straight lines on the hyperboloid passing through $(1,0,0)$; by Proposition 9.1.4, they are geodesics. The circle $z = 0, x^2+y^2 = 1$ and the hyperbola $y = 0, x^2 - z^2 = 1$ are both normal sections, hence geodesics by Proposition 9.1.6.

9.1.2 Let $\kappa(\boldsymbol{\gamma}) = \ddot{\boldsymbol{\gamma}}.(\mathbf{N}\times\dot{\boldsymbol{\gamma}})$. Note that if $\boldsymbol{\Gamma}(t) = \boldsymbol{\gamma}(\varphi(t))$ is a reparametrization of $\boldsymbol{\gamma}$, then $\kappa(\boldsymbol{\Gamma}) = \left(\frac{d\varphi}{dt}\right)^3 \kappa(\boldsymbol{\gamma})$. In particular, $\kappa(\boldsymbol{\gamma}) = 0 \iff \kappa(\boldsymbol{\Gamma}) = 0$.

For (i), let $\boldsymbol{\gamma}$ be a pre-geodesic and let $\boldsymbol{\Gamma}$ be a geodesic reparametrization of $\boldsymbol{\gamma}$. By Proposition 9.1.2 $\boldsymbol{\Gamma}$ has constant speed, say v, and then $\tilde{\boldsymbol{\Gamma}}(t) = \boldsymbol{\Gamma}(t/v)$ is a unit-speed geodesic. By Proposition 9.1.3, $\kappa(\tilde{\boldsymbol{\Gamma}}) = 0$, hence $\kappa(\boldsymbol{\Gamma}) = 0$, hence $\kappa(\boldsymbol{\gamma}) = 0$. Conversely, if $\kappa(\boldsymbol{\gamma}) = 0$ and if $\boldsymbol{\Gamma}$ is a unit-speed reparametrization of $\boldsymbol{\gamma}$, then $\kappa(\boldsymbol{\Gamma}) = 0$ so $\boldsymbol{\Gamma}$ is a geodesic by by Proposition 9.1.3.

Part (ii) is obvious.

For (iii), let $\boldsymbol{\gamma}$ be a constant speed pre-geodesic, say with speed v. Then $\boldsymbol{\Gamma}(t) = \boldsymbol{\gamma}(t/v)$ is a unit-speed pre-geodesic, hence a geodesic by (i) and Proposition 9.1.3. Since $\ddot{\boldsymbol{\gamma}} = v^2\ddot{\boldsymbol{\Gamma}}$, $\ddot{\boldsymbol{\gamma}}$ is perpendicular to the surface, so $\boldsymbol{\gamma}$ is a geodesic.

Finally, (iv) follows from (iii) and Proposition 9.1.2.

9.1.3 Let Π_s be the plane through $\boldsymbol{\gamma}(s)$ perpendicular to $\mathbf{t}(s)$; the parameter curve s = constant is the intersection of the surface with Π_s. From the solution to Exercise 4.2.7, the standard unit normal of $\boldsymbol{\sigma}$ is $\mathbf{N} = -(\cos\theta\,\mathbf{n} + \sin\theta\,\mathbf{b})$. Since this is perpendicular to $\mathbf{t}$, the circles in question are normal sections.

9.1.4 Take the ellipsoid to be $\frac{x^2}{p^2}+\frac{y^2}{q^2}+\frac{z^2}{r^2} = 1$; the vector $(\frac{x}{p^2}, \frac{y}{q^2}, \frac{z}{r^2})$ is normal to the ellipsoid by Exercise 5.1.2. If $\boldsymbol{\gamma}(t) = (f(t), g(t), h(t))$ is a curve on the ellipsoid, $R = (\frac{\dot{f}^2}{p^2}+\frac{\dot{g}^2}{q^2}+\frac{\dot{h}^2}{r^2})^{-1/2}$, $S = (\frac{f^2}{p^4}+\frac{g^2}{q^4}+\frac{h^2}{r^4})^{-1/2}$. Now, $\boldsymbol{\gamma}$ is a geodesic $\iff \ddot{\boldsymbol{\gamma}}$ is parallel to the normal $\iff (\ddot{f}, \ddot{g}, \ddot{h}) = \lambda(\frac{f}{p^2}, \frac{g}{q^2}, \frac{h}{r^2})$ for some scalar $\lambda(t)$. From $\frac{f^2}{p^2}+\frac{g^2}{q^2}+\frac{h^2}{r^2} = 1$ we get $\frac{f\dot{f}}{p^2}+\frac{g\dot{g}}{q^2}+\frac{h\dot{h}}{r^2} = 0$, hence $\frac{\dot{f}^2}{p^2}+\frac{\dot{g}^2}{q^2}+\frac{\dot{h}^2}{r^2}+\frac{f\ddot{f}}{p^2}+\frac{g\ddot{g}}{q^2}+\frac{h\ddot{h}}{r^2} = 0$, i.e.

$\frac{\dot{f}^2}{p^2}+\frac{\dot{g}^2}{q^2}+\frac{\dot{h}^2}{r^2}+\lambda\left(\frac{f^2}{p^4}+\frac{g^2}{q^4}+\frac{h^2}{r^4}\right)=0$, which gives $\lambda=-S^2/R^2$. The curvature $\parallel\ddot{\boldsymbol{\gamma}}\parallel=(\ddot{f}^2+\ddot{g}^2+\ddot{h}^2)^{1/2}=|\lambda|\left(\frac{f^2}{p^4}+\frac{g^2}{q^4}+\frac{h^2}{r^4}\right)^{1/2}=\frac{|\lambda|}{S}=\frac{S}{R^2}$. Finally,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\frac{1}{R^2S^2}\right) &= \left(\frac{f\dot{f}}{p^4}+\frac{g\dot{g}}{q^4}+\frac{h\dot{h}}{r^4}\right)\left(\frac{\dot{f}^2}{p^2}+\frac{\dot{g}^2}{q^2}+\frac{\dot{h}^2}{r^2}\right)\\ &\quad+\left(\frac{f^2}{p^4}+\frac{g^2}{q^4}+\frac{h^2}{r^4}\right)\left(\frac{\dot{f}\ddot{f}}{p^2}+\frac{\dot{g}\ddot{g}}{q^2}+\frac{\dot{h}\ddot{h}}{r^2}\right)\\ &= \frac{1}{R^2}\left(\frac{f\dot{f}}{p^4}+\frac{g\dot{g}}{q^4}+\frac{h\dot{h}}{r^4}\right)+\frac{\lambda}{S^2}\left(\frac{f\dot{f}}{p^4}+\frac{g\dot{g}}{q^4}+\frac{h\dot{h}}{r^4}\right)=0,\end{aligned}$$

since $\lambda=-S^2/R^2$. Hence, RS is constant.

9.1.5 Suppose that a geodesic $\boldsymbol{\gamma}$ lies in the plane $\mathbf{v.a}=b$, where $\mathbf{a}$ and b are constants. Then $\dot{\boldsymbol{\gamma}}.\mathbf{a}=\ddot{\boldsymbol{\gamma}}.\mathbf{a}=0$. Since $\ddot{\boldsymbol{\gamma}}$ is parallel to $\mathbf{N}$ (the unit normal of the surface), $\mathbf{N.a}=0$, so $\dot{\mathbf{N}}.\mathbf{a}=0$. Since $\mathbf{N}$, $\dot{\boldsymbol{\gamma}}$ and $\dot{\mathbf{N}}$ are all parallel to the plane and the last two vectors are perpendicular to the first, they are parallel. Hence $\boldsymbol{\gamma}$ is a line of curvature by Exercise 8.2.2. Conversely, if $\boldsymbol{\gamma}$ is both a geodesic and a line of curvature, we may assume $\boldsymbol{\gamma}$ has unit-speed (for the unit-speed reparametrization of $\boldsymbol{\gamma}$ would still be a geodesic and a line of curvature). Let $\mathbf{a}=\mathbf{N}\times\dot{\boldsymbol{\gamma}}$. Then $\dot{\mathbf{a}}=\dot{\mathbf{N}}\times\dot{\boldsymbol{\gamma}}+\mathbf{N}\times\ddot{\boldsymbol{\gamma}}=0$ since the first term vanishes because $\boldsymbol{\gamma}$ is a line of curvature and the second because $\boldsymbol{\gamma}$ is a geodesic. So $\mathbf{a}$ is constant. And $\dot{\boldsymbol{\gamma}}.\mathbf{a}=0$ so $\boldsymbol{\gamma}.\mathbf{a}$ is a constant, say b. Hence $\boldsymbol{\gamma}$ lies in the plane $\mathbf{v.a}=b$.

9.1.6 For (i) note that $\ddot{\boldsymbol{\gamma}}$ is a non-zero vector parallel to both $\mathbf{N}_1$ and $\mathbf{N}_2$, so $\mathbf{N}_1$ and $\mathbf{N}_2$ must be parallel. For an example, take $\mathcal{S}_1$ and $\mathcal{S}_2$ to be the sphere and cylinder in Theorem 6.4.6.
(ii) Now suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ intersect perpendicularly. Then, $\mathbf{N}_1,\mathbf{N}_2$ and $\dot{\boldsymbol{\gamma}}$ are perpendicular unit vectors. From $\dot{\boldsymbol{\gamma}}.\mathbf{N}_2=0$ we get $\ddot{\boldsymbol{\gamma}}.\mathbf{N}_2+\dot{\boldsymbol{\gamma}}.\dot{\mathbf{N}}_2=0$. If $\boldsymbol{\gamma}$ is a geodesic on $\mathcal{S}_1$, $\ddot{\boldsymbol{\gamma}}.\mathbf{N}_2=0$ since $\ddot{\boldsymbol{\gamma}}$ is parallel to $\mathbf{N}_1$, so $\dot{\mathbf{N}}_2$ is perpendicular to $\dot{\boldsymbol{\gamma}}$. Since $\dot{\mathbf{N}}_2$ is also perpendicular to $\mathbf{N}_2$, it must be parallel to $\mathbf{N}_1$. Conversely, if $\dot{\mathbf{N}}_2$ is parallel to $\mathbf{N}_1$, then $\dot{\boldsymbol{\gamma}}.\dot{\mathbf{N}}_2=0$ so $\ddot{\boldsymbol{\gamma}}$ is perpendicular to $\mathbf{N}_2$. Since $\ddot{\boldsymbol{\gamma}}$ is also perpendicular to $\dot{\boldsymbol{\gamma}}$, it must be parallel to $\mathbf{N}_1$. Finally, if $\boldsymbol{\gamma}$ is a geodesic on both $\mathcal{S}_1$ and $\mathcal{S}_2$, then $\dot{\mathbf{N}}_1$ is parallel to $\mathbf{N}_2$ and $\dot{\mathbf{N}}_2$ is parallel to $\mathbf{N}_1$. It follows that $(\mathbf{N}_1\times\mathbf{N}_2)^{\cdot}=\dot{\mathbf{N}}_1\times\mathbf{N}_2+\mathbf{N}_1\times\dot{\mathbf{N}}_2=\mathbf{0}$ so $\mathbf{N}_1\times\mathbf{N}_2$ is a constant vector. Since $\dot{\boldsymbol{\gamma}}$ is a unit vector parallel to $\mathbf{N}_1\times\mathbf{N}_2$, $\dot{\boldsymbol{\gamma}}$ is constant, so $\mathcal{C}$ is part of a straight line.

9.1.7 The ellipses formed by the intersections of the ellipsoid with the three coordinate planes are normal sections. Indeed, a normal to the ellipsoid is $\left(\frac{x}{p^2},\frac{y}{q^2},\frac{z}{r^2}\right)$. The plane $z=0$ (say) has normal $(0,0,1)$, so the intersection with the plane $z=0$ is a normal section if and only if $\left(\frac{x}{p^2},\frac{y}{q^2},\frac{z}{r^2}\right)\mathbf{.}(0,0,1)=z/r^2=0$ when $z=0$, which is true.

9.1.8 The intersections of the surface with each of the planes $x = \pm y$, $y = \pm z$ and $z = \pm x$ are normal sections. Indeed, by the solution to Exercise 8.2.37, a normal to the surface is $(x(y^2+z^2), y(x^2+z^2), z(x^2+y^2))$. The planes $x = \pm y$ have normals $(1, \mp 1, 0)$ so the intersection of the surface with $x = \pm y$ is a normal section if and only if

$$x(y^2+z^2) \mp y(x^2+z^2) = 0 \quad \text{when } x = \pm y,$$

which is evidently true. Similarly for the other four planes.

9.1.9 Assume that $\boldsymbol{\gamma}$ is unit-speed with tangent vector $\mathbf{t}$. Then (with the usual notation), $\mathbf{t.a}$ is constant, so $\dot{\mathbf{t}}.\mathbf{a} = 0$, i.e. $\kappa\mathbf{n.a} = 0$. Since $\kappa \neq 0$, $\mathbf{n}$ is perpendicular to $\mathbf{a}$. As $\boldsymbol{\gamma}$ is a geodesic, $\dot{\mathbf{t}}$ is parallel to $\mathbf{N}$, so $\mathbf{n}$ is parallel to $\mathbf{N}$. Hence, $\mathbf{a}$ is perpendicular to $\mathbf{N}$, hence tangent to $\mathcal{S}$.

9.1.10 From the solution to Exercise 7.3.3., a circle of radius r on a sphere of radius R has geodesic curvature $\kappa_g = \sqrt{R^2 - r^2}/rR$. Hence, $\kappa_g = 0 \iff r = R \iff$ the circle is a great circle.

9.1.11 The osculating plane is perpendicular to the binormal $\mathbf{b}$ of $\boldsymbol{\gamma}$, and so is perpendicular to $\mathcal{S}$ at $\mathbf{p}$ if and only if $\mathbf{b}$ is perpendicular to $\mathbf{N}$ at $\mathbf{p}$. Since $\mathbf{t}$ and $\mathbf{N}$ are perpendicular, $\mathbf{b}$ is perpendicular to $\mathbf{N} \iff \mathbf{n}$ is parallel to $\mathbf{N} \iff \boldsymbol{\gamma}$ is a geodesic.

9.1.12 $\boldsymbol{\gamma}$ is asymptotic $\iff \ddot{\boldsymbol{\gamma}}$ is perpendicular to $\mathbf{N}$; $\boldsymbol{\gamma}$ is a geodesic $\iff \ddot{\boldsymbol{\gamma}}$ is parallel to $\mathbf{N}$. Hence, if $\boldsymbol{\gamma}$ is both asymptotic and a geodesic, we must have $\ddot{\boldsymbol{\gamma}} = \mathbf{0}$, so $\boldsymbol{\gamma}(t) = t\mathbf{a} + \mathbf{b}$ for some constant vectors $\mathbf{a}, \mathbf{b}$, i.e. $\boldsymbol{\gamma}$ is a straight line.

9.1.13 $\boldsymbol{\gamma}$ is a geodesic on $\boldsymbol{\sigma} \iff \ddot{\boldsymbol{\gamma}}$ is parallel to $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (\dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}) \times \boldsymbol{\delta}$ when $v = 0$, i.e. $\ddot{\boldsymbol{\gamma}} \times (\dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta}) = \mathbf{0}$. Since $\ddot{\boldsymbol{\gamma}}$ is perpendicular to $\dot{\boldsymbol{\gamma}}$, $\boldsymbol{\gamma}$ is a geodesic $\iff \ddot{\boldsymbol{\gamma}}.\boldsymbol{\delta} = 0 \iff \boldsymbol{\delta}$ is perpendicular to $\ddot{\boldsymbol{\gamma}} = \kappa\mathbf{n}$.

9.1.14 Denoting by $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ the unit tangent vector, principal normal and binormal of $\boldsymbol{\Gamma}$, and by κ its curvature, we have $\mathbf{n} = \pm\mathbf{N}$ since $\boldsymbol{\Gamma}$ is a geodesic so

$$\dot{\boldsymbol{\Gamma}}.(\mathbf{N} \times \dot{\mathbf{N}}) = \mathbf{t}.(\mathbf{n} \times \dot{\mathbf{n}}) = \mathbf{t}.(\mathbf{n} \times (-\kappa\mathbf{t} + \tau\mathbf{b})) = \mathbf{t}.(\kappa\mathbf{b} + \tau\mathbf{t}) = \tau.$$

We can assume that $\boldsymbol{\gamma}$ is unit-speed; denote by a dash differentiation with respect to the arc-length of $\boldsymbol{\gamma}$. By Exercise 7.3.22, the geodesic torsion of $\boldsymbol{\gamma}$ is

$$\tau_g = (\boldsymbol{\gamma}' \times \mathbf{N}).\mathbf{N}' = \boldsymbol{\gamma}'.(\mathbf{N} \times \mathbf{N}').$$

Since $\boldsymbol{\gamma}$ and $\boldsymbol{\Gamma}$ touch at $\mathbf{p}$, $\boldsymbol{\gamma}' = \pm\dot{\boldsymbol{\Gamma}}$ at $\mathbf{p}$, and we can assume that the sign is $+$ by changing the sign of the parameter of $\boldsymbol{\gamma}$ if necessary. Since $\boldsymbol{\gamma}' = u'\boldsymbol{\sigma}_u + v'\boldsymbol{\sigma}_v$, $\mathbf{N}' = u'\mathbf{N}_u + v'\mathbf{N}_v$, etc., it follows that $u' = \dot{u}$ and $v' = \dot{v}$ at $\mathbf{p}$, and hence that $\mathbf{N}' = \dot{\mathbf{N}}$ at $\mathbf{p}$. Hence, $\tau_g = \dot{\boldsymbol{\Gamma}}.(\mathbf{N} \times \dot{\mathbf{N}})$ and the result now follows from the first part of the exercise.

Since $\mathbf{\Gamma}$ touches itself at each of its points (!), its torsion is equal to its geodesic torsion.

9.1.15 If $\boldsymbol{\gamma}$ is an asymptotic curve, then (with the usual notation) $\mathbf{b} = \pm\mathbf{N}$ so $\tau = -\dot{\mathbf{b}}.\mathbf{n} = \mp\dot{\mathbf{N}}.\mathbf{n}$. Since $\mathbf{n} = \mathbf{b}\times\mathbf{t} = \pm\mathbf{N}\times\mathbf{t}$, we get

$$\tau = -\dot{\mathbf{N}}.(\mathbf{N}\times\mathbf{t}) = \mathbf{t}.(\mathbf{N}\times\dot{\mathbf{N}}).$$

The last part follows from the preceding exercise.

9.1.16 We use the notation of Exercise 9.1.14 and assume that $\boldsymbol{\gamma}$ is a line of curvature. We can assume that $\dot{\mathbf{\Gamma}} = \boldsymbol{\gamma}'$ and $\mathbf{N}' = \dot{\mathbf{N}}$ at $\mathbf{p}$. By Rodrigues' formula, $\dot{\mathbf{N}} = -\kappa\dot{\boldsymbol{\gamma}}$ where κ is the principal curvature corresponding the line of curvature $\boldsymbol{\gamma}$, so $\mathbf{N}' = -\kappa\boldsymbol{\gamma}'$. Hence, the torsion of $\mathbf{\Gamma}$ is

$$\tau = \dot{\mathbf{\Gamma}}.(\mathbf{N}\times\dot{\mathbf{N}}) = -\kappa\boldsymbol{\gamma}'.(\mathbf{N}\times\boldsymbol{\gamma}') = 0.$$

9.1.17 (i) By Exercise 9.1.14, the torsion τ of a geodesic $\boldsymbol{\gamma}$ on a surface $\mathcal{S}$ is equal to its geodesic torsion which, by Exercise 8.2.22, is equal to $(\kappa_2 - \kappa_1)\sin\theta\cos\theta$ in the notation there. At an umbilic $\kappa_1 = \kappa_2$ so $\tau = 0$.
(ii) By (i), the torsion is given by $\tau = (\kappa_2 - \kappa_1)\sin\theta\cos\theta$. If a second geodesic $\tilde{\boldsymbol{\gamma}}$ intersects $\boldsymbol{\gamma}$ at right angles at $\mathbf{p}$, then in the obvious notation $\tilde{\theta} = \theta \pm \pi/2$ and hence $\tilde{\tau} = -\tau$.
(iii) If $\boldsymbol{\gamma}$ is a geodesic, its curvature κ is equal, up to sign, to its normal curvature, so $\kappa = \pm(\kappa_1\cos^2\theta + \kappa_2\sin^2\theta)$ by Euler's Theorem 8.2.4. Taking the $+$ sign,

$$\kappa - \kappa_1 = (\kappa_2 - \kappa_1)\sin^2\theta, \quad \kappa - \kappa_2 = (\kappa_1 - \kappa_2)\cos^2\theta,$$

so

$$(\kappa - \kappa_1)(\kappa - \kappa_2) = -(\kappa_1 - \kappa_2)^2\sin^2\theta\cos^2\theta = -\tau^2.$$

Taking the $-$ sign gives the other expression for τ^2.
(iv) If the surface is flat, either $\kappa_1 = 0$ or $\kappa_2 = 0$ at $\mathbf{p}$. If $\kappa_2 = 0$, then $\kappa = \pm\kappa_1\cos^2\theta$, $\tau = -\kappa_1\sin\theta\cos\theta$, so $\tau = \mp\kappa\tan\theta$. The other formula results if $\kappa_1 = 0$ at $\mathbf{p}$.

9.1.18 We can assume that $\boldsymbol{\gamma}$ is unit-speed as conditions (i) - (iii) are unchanged by reparametrization. Let $\boldsymbol{\delta}(u)$ be a non-zero vector parallel to the ruling through $\boldsymbol{\gamma}(u)$. Then (i) says that $\boldsymbol{\gamma}$ is a geodesic, and by Exercise 9.1.13 this holds $\iff \dot{\mathbf{t}}.\boldsymbol{\delta} = 0$. By Exercise 5.3.4, (ii) holds $\iff \mathbf{t}.\dot{\boldsymbol{\delta}} = 0$. And (iii) obviously holds $\iff \mathbf{t}.\boldsymbol{\delta}$ is a constant. Everything now follows immediately.

9.1.19 Suppose that every geodesic on a surface $\mathcal{S}$ is a plane curve. By Proposition 8.2.9, it suffices to show that every point of $\mathcal{S}$ is an umbilic. Suppose for a contradiction that $\mathbf{p} \in \mathcal{S}$ is not an umbilic, let κ_1, κ_2 be the distinct principal

curvatures of $\mathcal{S}$ at $\mathbf{p}$, and let $\mathbf{t}$ be a non-zero principal vector corresponding to the principal curvature κ_1 at $\mathbf{p}$. Choose an angle θ such that
(i) θ is not an integer multiple of $\pi/2$, and
(ii) $\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \neq 0$.

Let $\boldsymbol{\gamma}$ be the unit-speed geodesic on $\mathcal{S}$ passing through $\mathbf{p}$ and making an angle θ with $\mathbf{t}$. By condition (ii) and Euler's Theorem 8.4.2, $\boldsymbol{\gamma}$ has non-zero curvature at $\mathbf{p}$, and $\boldsymbol{\gamma}$ is a plane curve by the hypothesis. By Exercise 9.1.5, $\boldsymbol{\gamma}$ is a line of curvature, but this contradicts property (i).

9.1.20 Assume that $\boldsymbol{\gamma}$ is unit-speed, and let r be the radius of the sphere. Then, $(\boldsymbol{\gamma}-\mathbf{c}).(\boldsymbol{\gamma}-\mathbf{c}) = r^2$ so $\dot{\boldsymbol{\gamma}}.(\boldsymbol{\gamma}-\mathbf{c}) = 0$, $\ddot{\boldsymbol{\gamma}}.(\boldsymbol{\gamma}-\mathbf{c}) = -\dot{\boldsymbol{\gamma}}.\dot{\boldsymbol{\gamma}} = -1$. But, as $\boldsymbol{\gamma}$ is a geodesic, $\ddot{\boldsymbol{\gamma}} = \pm\kappa\mathbf{N}$, where κ is the curvature of $\boldsymbol{\gamma}$, so the last equation gives $\mathbf{N}.(\boldsymbol{\gamma}-\mathbf{c}) = \pm 1/\kappa$, i.e.

$$\kappa = \frac{1}{|\mathbf{N}.(\boldsymbol{\gamma}-\mathbf{c})|},$$

as required.

9.2.1 If $\mathbf{p}$ and $\mathbf{q}$ lie on the same parallel of the cylinder, there are exactly two geodesics joining them, namely the two circular arcs of the parallel of which $\mathbf{p}$ and $\mathbf{q}$ are the endpoints. If $\mathbf{p}$ and $\mathbf{q}$ are not on the same parallel, there are infinitely-many circular helices joining $\mathbf{p}$ and $\mathbf{q}$ (see Example 9.2.8).

9.2.2 Take the cone to be $\boldsymbol{\sigma}(u,v) = (u\cos v, u\sin v, u)$. By Exercise 6.2.1, $\boldsymbol{\sigma}$ is locally isometric to an open subset of the xy-plane by

$$\boldsymbol{\sigma}(u,v) \mapsto \left(u\sqrt{2}\cos\frac{v}{\sqrt{2}}, u\sqrt{2}\sin\frac{v}{\sqrt{2}}, 0\right).$$

By Corollary 9.2.7, the geodesics on the cone correspond to the straight lines in the xy-plane. Any such line, other than the axes $x = 0$ and $y = 0$, has equation $ax + by = 1$, where a, b are constants; this line corresponds to the curve

$$v \mapsto \left(\frac{\cos v}{\sqrt{2}(a\cos\frac{v}{\sqrt{2}} + b\sin\frac{v}{\sqrt{2}})}, \frac{\sin v}{\sqrt{2}(a\cos\frac{v}{\sqrt{2}} + b\sin\frac{v}{\sqrt{2}})}, \frac{1}{\sqrt{2}(a\cos\frac{v}{\sqrt{2}} + b\sin\frac{v}{\sqrt{2}})}\right);$$

the $x-$ and y-axes correspond to straight lines on the cone.

9.2.3 Parametrize the cylinder by $\boldsymbol{\sigma}(u,v) = (\cos u, \sin u, v)$. Then, $E = G = 1, F = 0$, so the geodesic equations are $\ddot{u} = \ddot{v} = 0$. Hence, $u = a + bt, v = c + dt$, where a, b, c, d are constants. If $b = 0$ this is a straight line on the cylinder; otherwise, it is a circular helix.

9.2.4 For the first part,

$$\begin{aligned}(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{\cdot} &= (E_u\dot{u} + E_v\dot{v})\,\dot{u}^2 + 2(F_u\dot{u} + F_v\dot{v})\dot{u}\dot{v} + (G_u\dot{u} + G_v\dot{v})\dot{v}^2 \\ &\qquad + 2E\dot{u}\ddot{u} + 2F(\dot{u}\ddot{v} + \ddot{u}\dot{v}) + 2G\dot{v}\ddot{v} \\ &= \dot{u}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2) + \dot{v}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2) \\ &\qquad + 2E\dot{u}\ddot{u} + 2F(\dot{u}\ddot{v} + \ddot{u}\dot{v}) + 2G\dot{v}\ddot{v} \\ &= 2(E_u\dot{u} + F\dot{v})^{\cdot}\dot{u} + 2(F\dot{u} + G\dot{v})^{\cdot}\dot{v} + 2(E\dot{u} + F\dot{v})\ddot{u} \\ &\qquad + 2(F\dot{u} + G\dot{v})\ddot{v} \ \text{ by the geodesic equations} \\ &= 2[(E\dot{u} + F\dot{v})\dot{u}]^{\cdot} + 2[(F\dot{u} + G\dot{v})\dot{v}]^{\cdot} \\ &= 2(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{\cdot}.\end{aligned}$$

Hence, $(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{\cdot} = 0$ and so $\parallel \dot{\boldsymbol{\gamma}} \parallel^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$ is constant. Suppose now that (i) and (ii) hold. Differentiating $E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 = \text{constant}$ gives

$$\begin{aligned}(E_u\dot{u} + E_v\dot{v})\dot{u}^2 + 2(F_u\dot{u} + F_v\dot{v}))\dot{u}\dot{v} + (G_u\dot{u} + G_v\dot{v})\dot{v}^2 \\ = -2(E\dot{u} + F\dot{v})\ddot{u} - 2(F\dot{u} + G\dot{v})\ddot{v},\end{aligned}$$

i.e.,

$$\begin{aligned}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)\dot{u} + (E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)\dot{v} \\ = -2(E\dot{u} + F\dot{v})\ddot{u} - 2(F\dot{u} + G\dot{v})\ddot{v}.\end{aligned}$$

Using (i) we get

$$2\dot{u}\frac{d}{dt}(E\dot{u} + F\dot{v}) + 2(E\dot{u} + F\dot{v})\ddot{v} = -2(F\dot{u} + G\dot{v})\ddot{v} - (E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)\dot{v}.$$

The left-hand side of this equation equals

$$2\frac{d}{dt}(\dot{u}(E\dot{u} + F\dot{v})) = -2\frac{d}{dt}(F\dot{u}\dot{v} + G\dot{v}^2) = -2(F\dot{u} + G\dot{v})\ddot{v} - 2\dot{v}\frac{d}{dt}(F\dot{u} + G\dot{v}).$$

Combining the last two equations gives (ii) provided $\dot{v} \neq 0$.

9.2.5 $E = 1, F = 0, G = 1+u^2$, so $\boldsymbol{\gamma}$ is unit-speed $\Longleftrightarrow \dot{u}^2 + (1+u^2)\dot{v}^2 = 1$. The second equation in (9.2) gives $\frac{d}{dt}((1+u^2)\dot{v}) = 0$, i.e. $\dot{v} = \frac{a}{1+u^2}$, where a is a constant. So $\dot{u}^2 = 1 - \frac{a^2}{1+u^2}$ and, along the geodesic, $\frac{dv}{du} = \frac{\dot{v}}{\dot{u}} = \pm\frac{a}{\sqrt{(1-a^2+u^2)(1+u^2)}}$. If $a = 0$, then $v = \text{constant}$ and we have a ruling. If $a = 1$, then $dv/du = \pm 1/u\sqrt{1+u^2}$, which can be integrated to give $v = v_0 \mp \sinh^{-1}\frac{1}{u}$, where v_0 is a constant.

For the last part, note that $\left(\frac{du}{dv}\right)^2 = \frac{(u^2+1)(u^2+1-a^2)}{a^2}$ so
(i) if $a^2 > 1$ then $du/dv = 0$ for $u = \pm\sqrt{a^2-1}$ and this is the minimum distance of the geodesic from the z-axis;
(ii) if $a^2 < 1$ then $|du/dv| > a^{-2} - 1$ so u will decrease to zero and the geodesic will cross the z-axis;
(iii) if $a^2 = 1$ then $du/dv = \pm(u^2+1)$ so $u = \pm\tan(v+c)$ where c is a constant.

The information given implies that, when $u = D$, $\cos\alpha = \dot{\boldsymbol{\gamma}}.\boldsymbol{\sigma}_u = \dot{u}$ (since $E = 1, F = 0$) so $a^2 = (1+D^2)\sin^2\alpha$. Then, a^2 is > 1, < 1 or $= 1$ according as D is $>$, $<$ or $= \cot\alpha$.

9.2.6 This is straightforward algebra.

9.2.7 By Exercise 5.3.3 we can parametrize the generalized cylinder $\mathcal{S}$ by $\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\delta}$, where $\boldsymbol{\gamma}$ is a unit-speed curve, $\boldsymbol{\delta}$ is a constant unit vector parallel to the rulings, and $\dot{\boldsymbol{\gamma}} = d\boldsymbol{\gamma}/du$ is perpendicular to $\boldsymbol{\delta}$ for all values of u. Then the first fundamental form of $\boldsymbol{\sigma}$ is $du^2 + dv^2$, so the map $\boldsymbol{\sigma}(u,v) \mapsto (u,v,0)$ is a local isometry from $\mathcal{S}$ to the xy-plane. The curves on $\mathcal{S}$ that make a constant angle with $\boldsymbol{\delta}$ correspond to the curves in the plane that make a constant angle with the axes. These are, of course, exactly the straight lines in the plane, i.e. the geodesics in the plane. It follows from Corollary 9.2.7 that the geodesics on $\mathcal{S}$ are exactly the curves on $\mathcal{S}$ making a constant angle with $\boldsymbol{\delta}$.

9.2.8 Let $\boldsymbol{\gamma}$ be a parameter curve v = constant, and assume that $\boldsymbol{\gamma}$ is unit-speed. Then, in the usual notation, $E\dot{u}^2 = 1$ so $\dot{u} = \pm 1/\sqrt{E}$. By Theorem 9.2.1, $\boldsymbol{\gamma}$ is a geodesic if and only if

$$(*) \qquad \frac{d}{dt}(E\dot{u}) = \frac{1}{2}E_u\dot{u}^2 \quad \text{and} \quad \frac{d}{dt}(F\dot{u}) = \frac{1}{2}E_v\dot{u}^2.$$

The second equation in (*) holds $\iff$

$$\left(\pm\frac{F}{\sqrt{E}}\right)_u \dot{u} = \frac{E_v}{2E},$$

i.e.

$$\frac{1}{\sqrt{E}}\left(\frac{F}{\sqrt{E}}\right)_u = \frac{E_v}{2E}.$$

This is equivalent to the stated condition. One checks in the same way that the first equation in (*) holds identically.

9.2.9 Let $\boldsymbol{\gamma}$ be the unit-speed curve on the surface corresponding to $u + v = c$, where c is a constant. Then, $\dot{u} + \dot{v} = 0$ so the unit-speed condition gives

$$(1 + u^2 + 2uv + 1 + v^2)\dot{u}^2 = 1,$$

i.e., $(2+c^2)\dot{u}^2 = 1$, so $\dot{u}$ (and hence also $\dot{v}$) is constant. The geodesic equations are

$$\frac{d}{dt}((1+u^2)\dot{u} - uv\dot{v}) = \dot{u}(u\dot{u} - v\dot{v}),$$
$$\frac{d}{dt}(-uv\dot{u} + (1+v^2)\dot{v}) = \dot{v}(v\dot{v} - u\dot{u}).$$

Using $\dot{v} = -\dot{u}$ the first equation becomes

$$\frac{d}{dt}((1+cu)\dot{u}) = c\dot{u}^2,$$

i.e. $(1+cu)\ddot{u} = 0$. Similarly, the second geodesic equation is equivalent to $(1+cv)\ddot{v} = 0$. Both equations are obviously satisfied when $\dot{u}$ and $\dot{v}$ are constant.

9.2.10 The geodesic equations

$$\ddot{u} = \frac{1}{2}G_u\dot{v}^2, \quad \frac{d}{dt}(G\dot{v}) = \frac{1}{2}G_v\dot{v}^2$$

must be satisfied if $v = cu$ and c is any constant. It is easy to see that these equations are satisfied if $c = 0$ (since in that case $\dot{u} = \pm 1$ from the unit-speed condition), so we assume that $c \neq 0$ from now on. Then the second equation is $\frac{d}{dt}(G\dot{u}) = \frac{1}{2}cG_v\dot{u}^2$, i.e.

$$G\ddot{u} + (G_u + \frac{1}{2}cG_v)\dot{u}^2 = 0.$$

Using the first geodesic equation and writing $c = v/u$ we get the stated equation for G.

Substituting $G = f(u)/v^2$ leads to the ordinary differential equation

$$(f + 2u^2)\frac{df}{du} = 2uf,$$

which can be solved for the function $f(u)$. (The solution is $2u^3 = 3f^3 \ln af$, where a is an arbitrary constant.)

9.2.11 Assume that the geodesic is unit-speed. The second geodesic equation in Theorem 9.2.1 gives $F\dot{u}+G\dot{v} = \Omega$, a constant. Using this and the unit-speed condition $E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 = 1$ gives

$$\dot{u} = \pm\sqrt{\frac{G - \Omega^2}{EG - F^2}}.$$

Hence, either $u =$ constant or, along the geodesic,

$$\frac{dv}{du} = \frac{\dot{v}}{\dot{u}} = \frac{\Omega - F\dot{u}}{G\dot{u}} = -\frac{F}{G} \pm \frac{\Omega}{G}\sqrt{\frac{EG - F^2}{G - \Omega^2}}.$$

9.2.12 The surface is parametrized by $\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\mathbf{b}(u)$, where $\mathbf{b}$ is the binormal of $\boldsymbol{\gamma}$ (we assume that $\boldsymbol{\gamma}$ is unit-speed). Then $\boldsymbol{\sigma}_u = \mathbf{t} - \tau v\mathbf{n}$, $\boldsymbol{\sigma}_v = \mathbf{b}$ and the first fundamental form is

$$(1+\tau v^2)du^2 + dv^2.$$

Let $\boldsymbol{\Gamma}(t)$ be a unit-speed geodesic on $\mathcal{S}$, and denote d/dt by a dot. The first geodesic equation in Theorem 9.2.1 gives

$$(1+\tau^2v^2)\dot{u} = c,$$

where c is a constant. To determine c, note that, at the intersection point $\mathbf{p}$ (say), $\dot{\boldsymbol{\Gamma}}.\mathbf{t} = \cos\alpha$, which gives

$$(\dot{u}(\mathbf{t} - \tau v\mathbf{n}) + \dot{v}\mathbf{b}).\mathbf{t} = \cos\alpha,$$

i.e. $\dot{u} = \cos\alpha$ at $\mathbf{p}$. Since $v = 0$ at $\mathbf{p}$, we have $c = \cos\alpha$ and so

$$\dot{u} = \frac{\cos\alpha}{1+\tau^2v^2}.$$

The unit-speed condition $(1+\tau^2v^2)\dot{u}^2 + \dot{v}^2 = 1$ now gives

$$\dot{v}^2 = \frac{\sin^2\alpha + \tau^2v^2}{1+\tau^2v^2},$$

and hence

$$\frac{du}{dv} = \pm\frac{\cos\alpha}{\sqrt{(\sin^2\alpha + \tau^2v^2)(1+\tau^2v^2)}}.$$

If $\tau \neq 0$, $|du/dv|$ decreases as $|v|$ increases and, for large $|v|$, $|du/dv|$ is approximately $\cos\alpha/\tau^2v^2$. It follows that u approaches limits $u_\pm$ (say) as $v \to \pm\infty$ and that u is always between these limits.

If τ is always zero, then $du/dv = \pm\cot\alpha$. If $\cot\alpha \neq 0$ then $u \to \pm\infty$ as $v \to \pm\infty$ and then $\boldsymbol{\gamma}$ is not contained between two rulings. If $\cot\alpha = 0$ then u is constant and $\boldsymbol{\gamma}$ is a ruling.

9.2.13 Writing $E = (U+V)P$, $G = (U+V)Q$, we have $\cos\theta = \dot{u}\sqrt{E}$, $\sin\theta = \dot{v}\sqrt{G}$ so we have to prove that $UG\dot{v}^2 - VE\dot{u}^2$ is constant along $\boldsymbol{\gamma}$. Now

$$2\frac{d}{dt}(UG\dot{v}^2 - VE\dot{u}^2) = 2U\dot{v}\frac{d}{dt}(G\dot{v}) - 2V\dot{u}\frac{d}{dt}(E\dot{u}) + 2G\dot{v}\frac{d}{dt}(U\dot{v}) - 2E\dot{u}\frac{d}{dt}(V\dot{u}).$$

Using the geodesic equations in Theorem 9.2.1, this becomes

$$
\begin{aligned}
&\dot{v}U(E_v\dot{u}^2 + G_v\dot{v}^2) - \dot{u}V(E_u\dot{u}^2 + G_u\dot{v}^2) + 2G\dot{v}\frac{d}{dt}(U\dot{v}) - 2E\dot{u}\frac{d}{dt}(V\dot{u})\\
&= \dot{u}^2\dot{v}(UE_v - 2EV_v) + \dot{u}\dot{v}^2(2GU_u - VG_u) + 2GU\dot{v}\ddot{v} - 2EV\dot{u}\ddot{u}\\
&\qquad\qquad + G_vU\dot{v}^3 - E_uV\dot{u}^3\\
&= \dot{u}^2\dot{v}(UE_v - 2EV_v) + \dot{u}\dot{v}^2(2GU_u - VG_u)\\
&\qquad + U\frac{d}{dt}(G\dot{v}^2) - UG_u\dot{u}\dot{v}^2 - V\frac{d}{dt}(E\dot{u}^2) + VE_v\dot{u}^2\dot{v}\\
&= \dot{u}^2\dot{v}((U+V)E_v - EV_v) + \dot{u}\dot{v}^2(GU_u - (U+V)G_u)\\
&\qquad + U\frac{d}{dt}(G\dot{v}^2) + U_uG\dot{u}\dot{v}^2 - V\frac{d}{dt}(E\dot{u}^2)V_vE\dot{u}^2\dot{v}\\
&= \frac{d}{dt}(UG\dot{v}^2 - VE\dot{u}^2) + \dot{u}^2\dot{v}((U+V)E_v - EV_v) + \dot{u}\dot{v}^2(GU_u - (U+V)G_u).
\end{aligned}
$$

Thus,

$$\frac{d}{dt}(UG\dot{v}^2 - VE\dot{u}^2) = \dot{u}^2\dot{v}((U+V)E_v - EV_v) + \dot{u}\dot{v}^2(GU_u - (U+V)G_u).$$

But, since U and P depend only on u, $E_v = V_vP$ and so $(U+V)E_v - EV_v = (U+V)(V_vP - V_vP) = 0$. Similarly, $GU_u - (U+V)G_u = 0$.

9.2.14 The first part can be verified directly. Alternatively, this parametrization can be deduced from that in (5.12) by putting $a = p^2 - u$, $b = q^2 - u$, $c = r^2 - u$ and then reparametrizing by $\tilde{u} = u - v$, $\tilde{v} = u - w$.

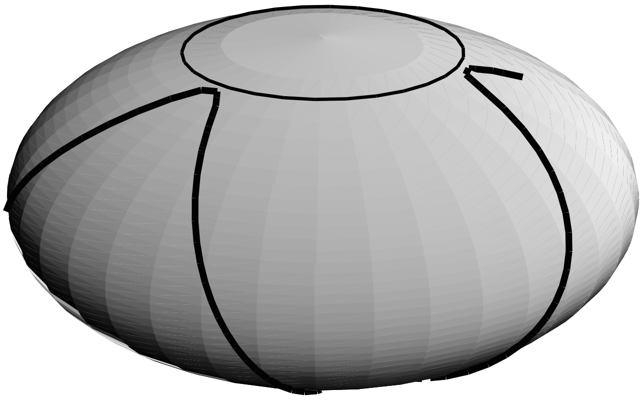
Straightforward algebra gives the first fundamental form as

$$(u - v)\left(\frac{udu^2}{(a+u)(b+u)(c+u)} - \frac{vdv^2}{(a+v)(b+v)(c+v)}\right).$$

Thus, the quadric is a Liouville surface with $U = u$, $V = -v$. The result now follows from the preceding exercise.

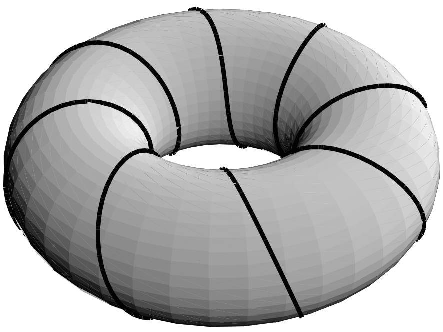
9.3.1 They are normal sections.

9.3.2 (i) Let the spheroid be obtained by rotating the ellipse $\frac{x^2}{p^2} + \frac{z^2}{q^2} = 1$ around the z-axis, where $p, q > 0$. Then, p is the maximum distance of a point of the spheroid from the z-axis, so the angular momentum Ω of a geodesic must be $\leq p$ (we can assume that $\Omega \geq 0$). If $\Omega = 0$, the geodesic is a meridian. If $0 < \Omega < a$, the geodesic is confined to the annular region on the spheroid contained between the circles $z = \pm q\sqrt{1 - \frac{\Omega^2}{p^2}}$, and the discussion in Example 9.3.3 shows that the geodesic 'bounces' between these two circles (see diagram below).

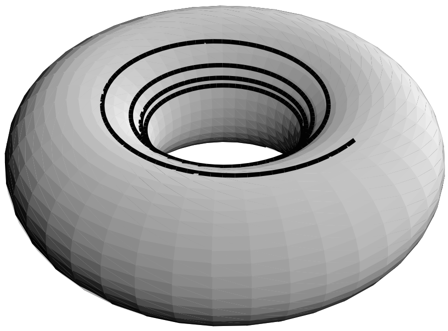


If $\Omega = p$, Eq. 9.10 shows that the geodesic must be the parallel $z = 0$.

(ii) Let the torus be as in Exercise 4.2.5. If $\Omega = 0$, the geodesic is a meridian (a circle). If $0 < \Omega < a - b$, the geodesic spirals around the torus. If $\Omega = a - b$, the geodesic is either the parallel of radius $a - b$ or spirals around the torus approaching this parallel asymptotically (but never crossing it):

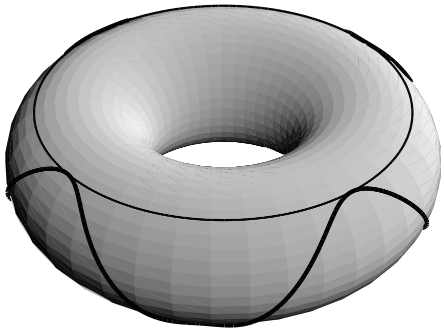


$0 < \Omega < a - b$



$\Omega = a - b$

If $a - b < \Omega < a + b$, the geodesic is confined to the annular region consisting of the part of the torus a distance $\geq \Omega$ from the axis, and bounces between the two parallels which bound this region:



If $\Omega = a + b$, the geodesic must be the parallel of radius $a + b$.

9.3.3 The two solutions of Eq. 9.14 are $v = v_0 \pm \sqrt{\frac{1}{\Omega^2} - w^2}$, so the condition for a self-intersection is that, for some $w > 1$, $2\sqrt{\frac{1}{\Omega^2} - w^2} = 2k\pi$ for some integer $k > 0$. This holds $\Longleftrightarrow 2\sqrt{\frac{1}{\Omega^2} - 1} > 2\pi$, i.e. $\Omega < (1+\pi^2)^{-1/2}$. In this case, there are k self-intersections, where k is the largest integer such that $2k\pi < 2\sqrt{\frac{1}{\Omega^2} - 1}$.

9.3.4 From the solution to Exercise 8.3.1, $Z = U + iV = \frac{z-i}{z+i}$, where $z = v + iw$. This is a Möbius transformation, so it takes lines and circles to lines and circles and preserves angles (Appendix 2). Since the geodesics on the pseudosphere correspond to straight lines and circles in the vw-plane perpendicular to the v-axis, they correspond in the VW-plane to straight lines and circles perpendicular to the image of the V-axis under the transformation $z \mapsto \frac{z-i}{z+i}$, i.e. the unit circle $V^2 + W^2 = 1$.

A straight line $a\bar{V} + b\bar{W} = c$ in the $\bar{V}\bar{W}$-plane (where a, b, c are constants) corresponds to the curve $2aV + 2bW = c(V^2 + W^2 + 1)$ in the VW-plane. If $c = 0$ this is a straight line through the origin, which corresponds to a geodesic on the pseudosphere by the first part. If $c \neq 0$ it is the equation of a circle with centre $(a/c, b/c)$ and squared radius $(a^2 + b^2 - c^2)/c^2$. This circle intersects the boundary circle $V^2 + W^2 = 1$ orthogonally because the square of the distance between the centres of the two circles is equal to the sum of the squares of their radii. Hence this circle also corresponds to a geodesic on the pseudosphere. This proves that every straight line in the $\bar{V}\bar{W}$-plane corresponds to a geodesic on the pseudosphere. That every geodesic on the pseudosphere arises from a straight line in the $\bar{V}\bar{W}$-plane in this way can be proved by similar arguments, or by noting that there is a straight line in the $\bar{V}\bar{W}$-plane passing through any point of the disc $\bar{V}^2 + \bar{W}^2 < 1$ in any direction and using Proposition 9.2.4.

9.3.5 By Proposition 9.3.1(ii), a surface of revolution for which every parallel is a geodesic would have $df/du = 0$ for all values of u, and hence $f =$ constant.

Then the surface would be a circular cylinder.

9.3.6 By Exercise 6.1.6(iii), the first fundamental form is $\cosh^2 u(du^2 + dv^2)$, so the result is a special case of Exercise 9.2.11.

9.3.7 By Clairaut's Theorem 9.3.2, ρ is constant along every geodesic. It follows that ρ is actually a constant (and hence the surface is a circular cylinder). For if not, then $d\rho/dz \neq 0$ for some value of z, say $z = z_0$. But then a geodesic passing through a point on the parallel $z = z_0$ and not tangent to that parallel would have non-constant ρ, contrary to hypothesis.

9.3.8 By Clairaut's Theorem, a geodesic on the unit cylinder (for which ρ is a constant) must have $\psi =$ constant, i.e. the geodesic must intersect the meridians (i.e. the rulings) of the cylinder at a constant angle. By Exercise 4.1.6, the curves on the cylinder that have this property are the straight lines (rulings), circles (parallels) and helices.

9.3.9 The first fundamental form is $du^2 + k^2 \cos^2 u dv^2$. By Clairaut's Theorem,

$$(k^2 \cos^2 u)\dot{v} = \Omega,$$

where Ω is a constant. As in Exercise 9.2.11, we find that, along any geodesic that is not a parallel,

$$\left(\frac{dv}{du}\right)^2 = \frac{\Omega^2}{k^2 \cos^2 u(k^2 \cos^2 u - \Omega^2)}.$$

The geodesic $\boldsymbol{\gamma}$ makes an angle α with the parallel $u = 0$ at the point $\boldsymbol{\sigma}(0,0) = (k,0,0)$. Hence,

$$\frac{\dot{\boldsymbol{\gamma}}.\boldsymbol{\sigma}_v}{k} = \cos\alpha,$$

which gives $\dot{v} = \cos\alpha/k$ at $(k,0,0)$. On the other hand, $\dot{v} = \Omega/k^2 \cos^2 u = \Omega/k^2$ at $(k,0,0)$. Hence, $\Omega = k\cos\alpha$ and we get

$$\left(\frac{dv}{du}\right)^2 = \frac{\cos^2\alpha}{k^2 \cos^2 u(\cos^2 u - \cos^2\alpha)}.$$

This gives

$$k\frac{dv}{du} = \pm\frac{\sec^2 u}{\sqrt{\tan^2\alpha - \tan^2 u}}.$$

Integrating,

$$kv = \pm\sin^{-1}\left(\frac{\tan u}{\tan\alpha}\right),$$

and hence $\tan u = \pm \tan\alpha \sin kv$. Since $|\sin kv| \leq 1$, the maximum value of u along the geodesic is α, so the maximum height above the xy-plane is

$$\int_0^\alpha \sqrt{1 - k^2 \sin^2\theta}\, d\theta.$$

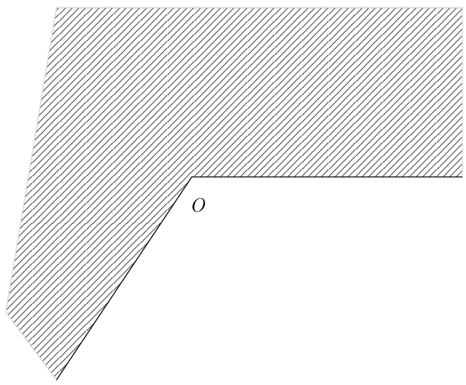
9.3.10 (i) We note that all geodesics $\boldsymbol{\gamma}(t)$ on the pseudosphere, except meridians, are defined only for t in some finite interval $\alpha \leq t \leq \beta$, say, whereas meridians are defined on a semi-infinite interval. Since $f(\boldsymbol{\gamma}(t))$ is a geodesic defined on the *same* interval as $\boldsymbol{\gamma}(t)$, it follows that f takes meridians to meridians, i.e. if v is constant, so is $\tilde{v}$. Hence, $\tilde{v}$ does not depend on w.

(ii) f preserves angles and takes meridians to meridians, so must take parallels to parallels. Hence, $\tilde{w}$ does not depend on v.

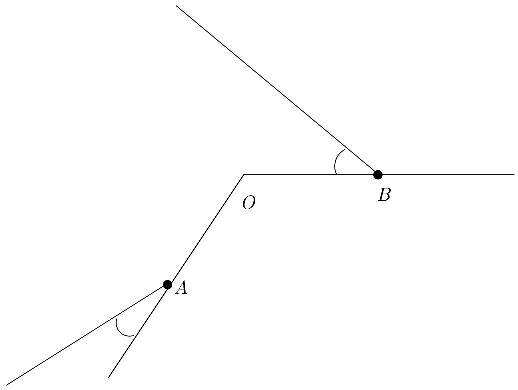
(iii) The parallel $w =$ constant has length $2\pi/w$ by Exercise 8.3.2(i) ($w = e^{-u}$). As f preserves lengths, part (ii) implies that $2\pi/w = 2\pi/\tilde{w}$, so $w = \tilde{w}$.

(iv) We now know that $f(\boldsymbol{\sigma}(u,v)) = \boldsymbol{\sigma}(F(v), w)$ for some smooth function $F(v)$. The first fundamental form of $\boldsymbol{\sigma}(F(v), w)$ is $w^{-2}\left(\left(\frac{dF}{dv}\right)^2 dv^2 + dw^2\right)$; since f is an isometry, this is equal to $w^{-2}(dv^2 + dw^2)$, hence $dF/dv = \pm 1$, so $F(v) = \pm v + \alpha$, where α is a constant. If the sign is $+$, f is rotation by α around the z-axis; if the sign is $-$, f is reflection in the plane containing the z-axis making an angle $\alpha/2$ with the xz-plane.

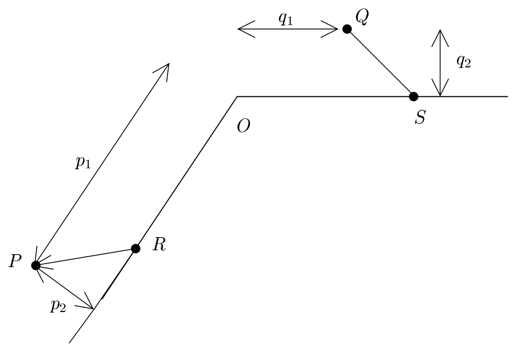
9.4.1 From Exercise 6.2.1, the cone is isometric to the 'sector' $\mathcal{S}$ of the plane with vertex at the origin and angle $\pi\sqrt{2}$:



Geodesics on the cone correspond to possibly broken line segments in $\mathcal{S}$: if a line segment meets the boundary of $\mathcal{S}$ at a point A, say, it may continue from the point B on the other boundary line at the same distance as A from the origin and with the indicated angles being equal:

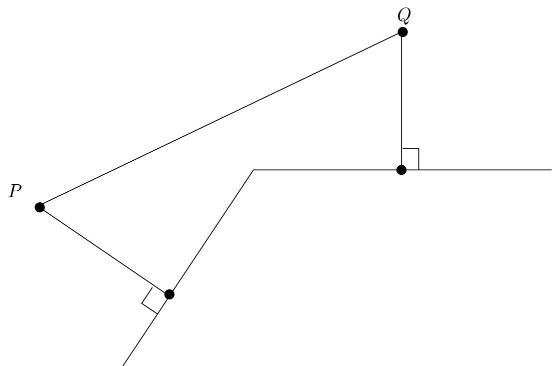


(i) TRUE: if two points P and Q can be joined by a line segment in $\mathcal{S}$ there is no problem; otherwise, P and Q can be joined by a broken line segment satisfying the conditions above:



To see that this is always possible, let p_1, p_2, q_1 and q_2 be the indicated distances, and let R and S be the points on the boundary of the sector at a distance $(p_2 q_1 + p_1 q_2)/(p_2 + q_2)$ from the origin. Then, the broken line segment joining P and R followed by that joining S and Q is the desired geodesic.

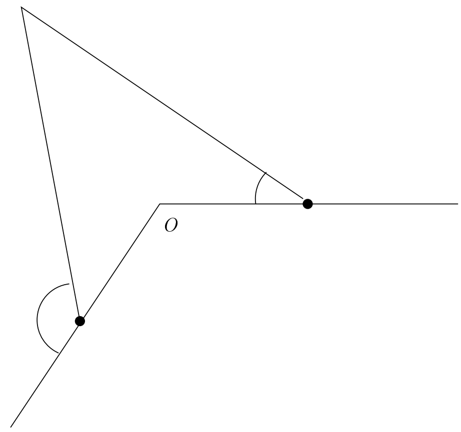
(ii) FALSE:



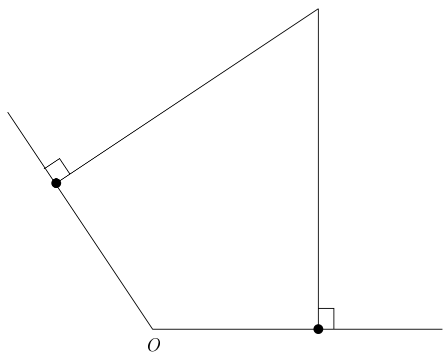
(iii) FALSE: many meet in two points, such as the two geodesics joining P and Q in the diagram in (ii).

(iv) TRUE: the meridians do not intersect (remember that the vertex of the cone has been removed), and parallel straight lines that are entirely contained in $\mathcal{S}$

do not intersect.
(v) TRUE: since (broken) line segments in $\mathcal{S}$ can clearly be continued indefinitely in both directions.
(vi) TRUE: a situation of the form



in which the indicated angles are equal is clearly impossible. But the answer to this part of the question depends on the angle of the cone: if the angle is α, instead of $\pi/4$, lines can self-intersect if $\alpha < \pi/6$, for then the corresponding sector in the plane has angle $< \pi$:



9.4.2 We consider the intersection of S^2 with the plane passing through $\mathbf{p}$ and $\mathbf{q}$ and making an angle θ with the xy-plane, where $-\pi/2 < \theta < \pi/2$. This intersection is a circle $\mathcal{C}_\theta$ but it is not a great circle unless $\theta = 0$. Hence, if $\theta \neq 0$, the short segment of $\mathcal{C}_\theta$ joining $\mathbf{p}$ and $\mathbf{q}$ is not a geodesic and so has length $> \pi/2$ (the length of the shortest geodesic joining $\mathbf{p}$ and $\mathbf{q}$). Since the length of $\mathcal{C}_\theta$ is $\leq 2\pi$, the length of the long segment of $\mathcal{C}_\theta$ joining $\mathbf{p}$ and $\mathbf{q}$ has length $< 3\pi/2$ if $\theta \neq 0$, i.e. strictly less than the length of the long segment of the geodesic $\mathcal{C}_0$ joining $\mathbf{p}$ and $\mathbf{q}$. So the long geodesic segment is not a local minimum of the length of curves joining $\mathbf{p}$ and $\mathbf{q}$.

9.4.3 (i) This is obvious if $n \geq 0$ since $e^{-1/t^2} \to 0$ as $t \to 0$. We prove that $t^{-n}e^{-1/t^2} \to 0$ as $t \to 0$ by induction on $n \geq 0$. We know the result if $n = 0$, and if $n > 0$

we can apply L'Hopital's rule: $\lim_{t\to 0}\frac{t^{-n}}{e^{1/t^2}} = \lim_{t\to 0}\frac{nt^{-n-1}}{\frac{2}{t^3}e^{1/t^2}} = \lim_{t\to 0}\frac{n}{2}\frac{t^{-(n-2)}}{e^{1/t^2}}$, which vanishes by the induction hypothesis.

(ii) We prove by induction on n that θ is n-times differentiable with

$$\frac{d^n\theta}{dt^n} = \begin{cases} \frac{P_n(t)}{t^{3n}}e^{-1/t^2} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0, \end{cases}$$

where P_n is a polynomial in t. For $n = 0$, the assertion holds with $P_0 = 1$. Assuming the result for some $n \geq 0$, $\frac{d^{n+1}\theta}{dt^{n+1}} = \left(\frac{-3nP_n}{t^{3n+1}} + \frac{P_n'}{t^{3n}} + \frac{2P_n}{t^{3n+3}}\right)e^{-1/t^2}$ if $t \neq 0$, so we take $P_{n+1} = (2-3nt^2)P_n + t^3P_n'$. If $t = 0$, $\frac{d^{n+1}\theta}{dt^{n+1}} = \lim_{t\to 0}\frac{P_n(t)}{t^{3n+1}}e^{-1/t^2} = P_n(0)\lim_{t\to 0}\frac{e^{-1/t^2}}{t^{3n+1}} = 0$ by part (i). Parts (iii) and (iv) are obvious.

9.4.4 For the circular cylinder, the answers are: (i) true; (ii) false; (iii) false; (iv) true; (v) true; (vi) false; (vii) true.

For the sphere: (i) true; (ii) false; (iii) false; (iv) false; (v) true; (vi) false; (vii) true.

9.5.1 Since $\boldsymbol{\gamma}^\theta$ is unit-speed, $\boldsymbol{\sigma}_r.\boldsymbol{\sigma}_r = 1$, so $\int_0^R \boldsymbol{\sigma}_r.\boldsymbol{\sigma}_r\,dr = R$. Differentiating with respect to θ gives $\int_0^R \boldsymbol{\sigma}_r.\boldsymbol{\sigma}_{r\theta}\,dr = 0$, and then integrating by parts gives

$$\boldsymbol{\sigma}_\theta.\boldsymbol{\sigma}_r\big|_{r=0}^{r=R} - \int_0^R \boldsymbol{\sigma}_\theta.\boldsymbol{\sigma}_{rr}\,dr = 0.$$

Now $\boldsymbol{\sigma}(0,\theta) = \mathbf{p}$ for all θ, so $\boldsymbol{\sigma}_\theta = \mathbf{0}$ when $r = 0$. So we must show that the integral in the last equation vanishes. But, $\boldsymbol{\sigma}_{rr} = \ddot{\boldsymbol{\gamma}}^\theta$, the dot denoting the derivative with respect to the parameter r of the geodesic $\boldsymbol{\gamma}^\theta$, so $\boldsymbol{\sigma}_{rr}$ is parallel to the unit normal $\mathbf{N}$ of $\boldsymbol{\sigma}$; since $\boldsymbol{\sigma}_\theta.\mathbf{N} = 0$, it follows that $\boldsymbol{\sigma}_\theta.\boldsymbol{\sigma}_{rr} = 0$. The first fundamental form is as indicated since $\boldsymbol{\sigma}_r.\boldsymbol{\sigma}_r = 1$ and $\boldsymbol{\sigma}_r.\boldsymbol{\sigma}_\theta = 0$.

9.5.2 (i) The length of the part of $\boldsymbol{\gamma}$ between $\mathbf{p}$ and $\mathbf{q}$ is $\int_0^1\sqrt{\dot f^2 + G\dot g^2}dt \geq \int_0^1\sqrt{\dot f^2}dt = f(1) - f(0) = R$.

(ii) Use the hint, noting that the length of the part of $\boldsymbol{\gamma}$ between $\mathbf{p}$ and $\mathbf{q}'$ is $\geq R$.

(iii) If the part of $\boldsymbol{\gamma}$ between $\mathbf{p}$ and $\mathbf{q}$ has length R, then it must stay inside the geodesic circle with centre $\mathbf{p}$ and radius R by (ii), and then we must have $\int_0^1\sqrt{\dot f^2 + G\dot g^2}dt = \int_0^1\sqrt{\dot f^2}dt$. Then $G\dot g = 0$ for all $t \in (0,1)$, so $\dot g = 0$ (as $G > 0$) and so g is a constant which must be α as $\boldsymbol{\gamma}$ passes through $\mathbf{q}$. This means that $\boldsymbol{\gamma}$ is a parametrization of the radial line $\theta = \alpha$.

9.5.3 The first fundamental form is $dr^2 + Gd\theta^2$ and a geodesic circle is a parameter curve $r =$ constant. By Exercise 7.3.15, its geodesic curvature is $G_r/2G$. We are

given that this is a function of r only, say $A(r)$. Then,

$$\frac{\partial}{\partial r}(\ln G) = A(r)$$
$$\therefore \quad \ln G = \int A(r)dr + B(\theta) \quad \text{(say)}$$
$$\therefore \quad G = f(r)g(\theta),$$

where $f(r) = e^{\int A(r)dr}$ and $g(\theta) = e^{B(\theta)}$.

Chapter 10

10.1.1 The matrix of the Weingarten map with respect to the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ is $\mathcal{F}_I^{-1}\mathcal{F}_{II} = = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -\cos^2 v & 0 \\ 0 & -1 \end{pmatrix} = -I$, so $\mathbf{N}_u = \boldsymbol{\sigma}_u$, $\mathbf{N}_v = \boldsymbol{\sigma}_v$. Thus, $\mathbf{N} = \boldsymbol{\sigma} - \mathbf{a}$, where $\mathbf{a}$ is a constant vector. Hence, $\parallel \boldsymbol{\sigma} - \mathbf{a} \parallel = 1$, showing that the surface is an open subset of the sphere $\mathcal{S}$ of radius 1 and centre $\mathbf{a}$. The standard latitude-longitude parametrization $\boldsymbol{\sigma}(u, v)$ of S^2 has first and second fundamental forms both given by $du^2 + \cos^2 u dv^2$, so the parametrization $\boldsymbol{\sigma}(v, u) + \mathbf{a}$ of $\mathcal{S}$ has the given first and second fundamental forms (the second fundamental form changes sign because $\boldsymbol{\sigma}_v \times \boldsymbol{\sigma}_u = -\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v$).

10.1.2 $\Gamma_{22}^1 = \sin u \cos u$ and the other Christoffel symbols are zero; the second Codazzi–Mainardi equation is not satisfied.

10.1.3 The Christoffel symbols are $\Gamma_{11}^1 = 0$, $\Gamma_{11}^2 = 1/w$, $\Gamma_{12}^1 = -1/w$, $\Gamma_{12}^2 = 0$, $\Gamma_{22}^1 = 0$, $\Gamma_{22}^2 = -1/w$. Using the first equation in Proposition 10.1.2 we get $K = -1$. The Codazzi-Mainardi equations are $L_w = -(L+N)/w$, $N_v = 0$. Hence, N depends only on w, and since $-1 = K = LN/EG$, we have $LN = -1/w^4$ so L also depends only on w; the first Codazzi-Mainardi equation gives $dL/dw = -L/w + 1/Lw^5$, which is the stated differential equation. Putting $P = Lw^2$ we get $dP/dw = (1+P^2)/wP$ which integrates to give $1 + P^2 = Cw^2$, where $C > 0$ is a constant, i.e. $L = \pm\sqrt{Cw^2 - 1}/w$. Hence, the second fundamental form is only defined for $w \geq C^{-1/2}$ or $w \leq -C^{-1/2}$.

The first fundamental form is this exercise is the same as that of a suitable parametrization of the pseudosphere (see Exercise 8.3.1(i)). We saw that the pseudosphere corresponds to (part of) the region $w > 1$.

10.1.4 The Christoffel symbols are $\Gamma_{11}^1 = E_u/2E$, $\Gamma_{11}^2 = -E_v/2G$, $\Gamma_{12}^1 = E_v/2E$, $\Gamma_{12}^2 = G_u/2G$, $\Gamma_{22}^1 = -G_u/2E$, $\Gamma_{22}^2 = G_v/2G$. The first Codazzi–Mainardi equation is

$$L_v = \frac{LE_v}{2E} - N\left(\frac{-E_v}{2G}\right) = \frac{1}{2}E_v\left(\frac{L}{E} + \frac{N}{G}\right),$$

and similarly for the other equation. Finally,

$$(\kappa_1)_v = \frac{E_v}{2E}\left(\frac{L}{E}+\frac{N}{G}\right) - \frac{LE_v}{E^2} = \frac{E_v}{2E}\left(\frac{N}{G}-\frac{L}{E}\right) = \frac{E_v}{2E}(\kappa_2-\kappa_1),$$

and similarly for $(\kappa_2)_u$.

10.1.5 If E, F, G, L, M, N are constant, the Christoffel symbols are all zero so Eqs. 10.1 are obviously satisfied, and the Gauss equations (Proposition 10.1.2) are satisfied $\iff K = 0$, i.e. $LN = M^2$. Of course, we must also have $E > 0, G > 0$ and $EG - F^2 > 0$.

Let C be the constant matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and let $\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$, $\mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$. By Exercises 6.1.4 and 7.1.3, the reparametrization $(u, v) \mapsto (\tilde{u}, \tilde{v})$ leads to the stated first and second fundamental forms if and only if

$$C^t\mathcal{F}_I C = I, \quad C^t\mathcal{F}_{II}C = \begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix}.$$

To prove the existence of an invertible matrix C satisfying these conditions, note first that since $\mathcal{F}_I$ is symmetric, there is an orthogonal matrix P such that $P^t\mathcal{F}_I P = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where λ, μ are the eigenvalues of $\mathcal{F}_I$ (see Appendix 0). Since the trace $E + G$ and the determinant $EG - F^2$ of $\mathcal{F}_I$ are both > 0, it follows that the sum and product of λ, μ are both > 0, and hence that λ, μ are both > 0. Letting $A = \begin{pmatrix} \lambda^{-1/2} & 0 \\ 0 & \mu^{-1/2} \end{pmatrix}$, we have $A^tP^t\mathcal{F}_I PA = I$. The matrix $A^tP^t\mathcal{F}_{II}PA$ is symmetric, so there is an orthogonal matrix Q such that $Q^t(A^tP^t\mathcal{F}_{II}PA)Q = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$, for some (constants) κ_1, κ_2. The matrix $C = PAQ$ has the property that $C^t\mathcal{F}_I C = I$, $C^t\mathcal{F}_{II}C = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$.

By Proposition 8.2.6, the principal curvatures are the roots of

$$\det(\mathcal{F}_{II} - \kappa\mathcal{F}_I) = 0.$$

Since C is invertible, these are the same as the roots of

$$\det(C^t\mathcal{F}_{II}C - \kappa C^t\mathcal{F}_I C) = 0,$$

i.e. κ_1, κ_2. Since $K = \kappa_1\kappa_2$, one of κ_1, κ_2 is zero, and we might as well assume that $\kappa_2 = 0$ (if $\kappa_1 = 0$, we can replace C by $C\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). This gives the stated result, with $\kappa_1 = \kappa$.

If $\kappa = 0$, the surface has the same first and second fundamental forms as a plane, so by Theorem 10.1.3 the surface can be obtained from a plane by applying a direct isometry of $\mathbb{R}^3$. It follows that the surface is a plane.

If $\kappa \neq 0$, then by the reparametrization $v \mapsto |\kappa|^{-1/2}v$ we can assume that $\kappa = \pm 1$. By Examples 6.1.4, 7.1.2 and 10.1.4, in both cases there is a parametrization of the unit cylinder with these first and second fundamental forms. By Theorem 10.1.3 again, the surface is obtained by applying to this cylinder a direct isometry of $\mathbb{R}^3$, and so is a circular cylinder.

10.1.6 (i) The Christoffel symbols are $\Gamma^1_{11} = \frac{E_u}{2E}$, $\Gamma^2_{11} = -\frac{E_v}{2G}$, $\Gamma^1_{12} = \frac{E_v}{2E}$, $\Gamma^2_{12} = \frac{G_u}{2G}$, $\Gamma^1_{22} = -\frac{G_u}{2E}$, $\Gamma^2_{22} = \frac{G_v}{2G}$. Eqs. 10.1 (with $L = N = 0, M = 1$) now state that $(E/G)_u = 0$ and $(E/G)_v = 0$. Hence, E/G is constant.
(ii) If $E/G = \lambda^4$, the reparametrization $\tilde{u} = \lambda u$, $\tilde{v} = \lambda^{-1}v$ leads to the first and second fundamental forms $\tilde{E}(d\tilde{u}^2 + d\tilde{v}^2)$ and $2d\tilde{u}d\tilde{v}$, respectively, where $\tilde{E} = \lambda^{-2}E$.
(iii) If $E = G$, the first of the Gauss equations in Proposition 10.1.2 gives

$$-\frac{1}{E} = -\left(\frac{E_v}{2E}\right)_v - \left(\frac{E_u}{2E}\right)_u + \left(\frac{E_u}{2E}\right)^2 - \left(\frac{E_v}{2E}\right)^2 + \left(\frac{E_v}{2E}\right)^2 - \left(\frac{E_u}{2E}\right)^2,$$

i.e.

$$\left(\frac{E_u}{2E}\right)_u + \left(\frac{E_v}{2E}\right)_v = \frac{2}{E},$$

which is the stated equation.

10.1.7 The assumption is that $L = N = 0$. The Codazzi-Mainardi equations are

$$-M_u = M(\Gamma^2_{12} - \Gamma^1_{11}), \quad M_v = M(\Gamma^2_{22} - \Gamma^1_{12}).$$

10.1.8 (i) By Proposition 8.4.1, there is a patch $\boldsymbol{\sigma}(U, V)$ of $\mathcal{S}$ containing $\mathbf{p}$ such that $F = M = 0$. If $\kappa_1 = L/E$ is a constant κ, Exercise 10.1.4 implies that $E_V = 0$ (since the principal curvatures are distinct by assumption). So E is a function of U only. Define $u = \int \sqrt{E}\, dU$, $v = V$. Then the first and second fundamental forms are of the form $du^2 + Gdv^2$ and $\kappa du^2 + Ndv^2$, respectively.
(ii) The Christoffel symbols are $\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{12} = 0$, $\Gamma^2_{12} = G_u/2G$, $\Gamma^1_{22} = -G_u/2G$, $\Gamma^2_{22} = G_v/2G$.
(iii) The parameter curve $v =$ constant is parametrized by $\boldsymbol{\Gamma}(u) = \boldsymbol{\sigma}(u, v)$. Denoting d/du by a dot, we have $\dot{\boldsymbol{\Gamma}} = \boldsymbol{\sigma}_u$, $\ddot{\boldsymbol{\Gamma}} = \boldsymbol{\sigma}_{uu} = \kappa \mathbf{N}$, by Proposition 7.4.4 and part (ii). By the proof of Proposition 8.1.2, $\dddot{\boldsymbol{\Gamma}} = \kappa \mathbf{N}_u = -\kappa^2 \boldsymbol{\sigma}_u$. Since $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v/\sqrt{G}, \mathbf{N}\}$ is a right-handed orthonormal basis of $\mathbb{R}^3$,

$$\frac{\parallel \dot{\boldsymbol{\Gamma}} \times \ddot{\boldsymbol{\Gamma}} \parallel}{\parallel \dot{\boldsymbol{\Gamma}} \parallel^3} = \parallel \frac{\kappa}{\sqrt{G}} \boldsymbol{\sigma}_v \parallel = |\kappa|.$$

The torsion of $\mathbf{\Gamma}$ is

$$\frac{\dddot{\mathbf{\Gamma}}.(\dot{\mathbf{\Gamma}} \times \ddot{\mathbf{\Gamma}})}{\| \dot{\mathbf{\Gamma}} \times \ddot{\mathbf{\Gamma}} \|^2} = \frac{(-\kappa^2\boldsymbol{\sigma}_u).(-\kappa\boldsymbol{\sigma}_v/\sqrt{G})}{\kappa^2} = 0,$$

since $\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_v = 0$. It follows that $\mathbf{\Gamma}$ is a circle of radius $r = 1/|\kappa|$.
(iv) $\boldsymbol{\sigma}_{uuu} = \kappa\mathbf{N}_u = -\kappa^2\boldsymbol{\sigma}_u$, so $\boldsymbol{\sigma}_{uu} + \frac{1}{r^2}u$ is a function of v only. Hence, $\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(v) + \mathbf{C}(v)\cos\frac{u}{r} + \mathbf{D}(v)\sin\frac{u}{r}$ for some smooth functions $\mathbf{C}, \mathbf{D}, \boldsymbol{\gamma}$ of v. We may assume that the curve $\boldsymbol{\gamma}$ is unit-speed, since replacing v by a smooth function of v does not change the coefficients E, F, L, M (as $F = M = 0$). By (iii), $\| \boldsymbol{\sigma}(u,v) \|^2 = r^2$ for all u, v. This implies that $\mathbf{C}$ and $\mathbf{D}$ are perpendicular vectors of length r. Let $\mathbf{c} = r^{-1}\mathbf{C}$, $\mathbf{d} = r^{-1}\mathbf{D}$.
(v) Using the fact that $\mathbf{c}(v)$ and $\mathbf{d}(v)$ are perpendicular unit vectors for all values of v, we get

$$\boldsymbol{\sigma}_u.\boldsymbol{\sigma}_v = \frac{d\boldsymbol{\gamma}}{dv}.\left(-\mathbf{c}\sin\frac{u}{r} + \mathbf{d}\cos\frac{u}{r}\right) + r\frac{d\mathbf{c}}{dv}.\mathbf{d}$$

for all u, v. Hence, $\frac{d\boldsymbol{\gamma}}{dv}.\mathbf{c} = \frac{d\boldsymbol{\gamma}}{dv}.\mathbf{d} = 0$ $(= \frac{d\mathbf{c}}{dv}.\mathbf{d})$. It follows that, for some angle α (possibly depending on v), we have

$$\mathbf{c} = \mathbf{n}\cos\alpha + \mathbf{b}\sin\alpha, \quad \pm\mathbf{d} = -\mathbf{n}\sin\alpha + \mathbf{b}\cos\alpha,$$

where $\mathbf{n}$ and $\mathbf{b}$ denote the principal normal and binormal of $\boldsymbol{\gamma}$, respectively. (The condition $\frac{d\boldsymbol{\gamma}}{dv}.\mathbf{d} = 0$ implies that $d\alpha/dv = -\tau$, where τ is the torsion of $\boldsymbol{\gamma}$.) Writing $s = v$, $\theta = \alpha \pm \kappa u$ shows that $\boldsymbol{\sigma}$ is a reparametrization of the tube of radius r around $\boldsymbol{\gamma}$ (Exercise 4.2.7).

10.1.9 Since $p^{-1}\mathbf{\Sigma}_u$, $q^{-1}\mathbf{\Sigma}_v$ and $r^{-1}\mathbf{\Sigma}_w$ are perpendicular unit vectors,

$$\mathbf{\Sigma}_{uu} = a\mathbf{\Sigma}_u + b\mathbf{\Sigma}_v + c\mathbf{\Sigma}_w,$$

for some scalars a, b, c. We have

$$p^2 a = \mathbf{\Sigma}_{uu}.\mathbf{\Sigma}_u = \frac{1}{2}(\mathbf{\Sigma}_u.\mathbf{\Sigma}_u)_u = \frac{1}{2}(p^2)_u = pp_u,$$

so $a = p_u/p$. Next,

$$q^2 b = \mathbf{\Sigma}_{uu}.\mathbf{\Sigma}_v = (\mathbf{\Sigma}_u.\mathbf{\Sigma}_v)_u - \mathbf{\Sigma}_u.\mathbf{\Sigma}_{uv} = -\frac{1}{2}(\mathbf{\Sigma}_u.\mathbf{\Sigma}_u)_v = -pp_v,$$

so $b = -pp_v/q^2$. Similarly, $c = -pp_w/r^2$, hence the stated formula for $\mathbf{\Sigma}_{uu}$. The other formulas for the second derivatives of $\mathbf{\Sigma}$ are proved similarly.

We now use these formulas to compute both sides of the equation $(\mathbf{\Sigma}_{uu})_v = (\mathbf{\Sigma}_{uv})_u$; both sides are then linear combinations of $\mathbf{\Sigma}_u, \mathbf{\Sigma}_v$ and $\mathbf{\Sigma}_w$. We find that

the coefficients of $\boldsymbol{\Sigma}_u$ are automatically equal, while equating coefficients of $\boldsymbol{\Sigma}_v$ and $\boldsymbol{\Sigma}_w$ on the two sides yields the third and fourth of Lamé's equations. The others are proved similarly.

10.2.1 By Corollary 10.2.3(i),

$$K = -\frac{1}{2e^\lambda}\left(\frac{\partial}{\partial u}\left(\frac{(e^\lambda)_u}{e^\lambda}\right) + \frac{\partial}{\partial v}\left(\frac{(e^\lambda)_v}{e^\lambda}\right)\right) = -\frac{1}{2e^\lambda}(\lambda_{uu} + \lambda_{vv}).$$

10.2.2 By Exercise 6.1.4, $\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = J^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} J$, where $J = \begin{pmatrix} \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \\ \frac{\partial \theta}{\partial u} & \frac{\partial \theta}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{u}{r} & \frac{v}{r} \\ \frac{-v}{r^2} & \frac{u}{r^2} \end{pmatrix}$. By Exercise 9.5.1, $E = 1, F = 0$, and we get the stated formulas for $\tilde{E}, \tilde{F}, \tilde{G}$. From $\tilde{E} - 1 = \frac{v^2}{r^2}\left(\frac{G}{r^2} - 1\right)$, $\tilde{G} - 1 = \frac{u^2}{r^2}\left(\frac{G}{r^2} - 1\right)$, we get $u^2(\tilde{E} - 1) = v^2(\tilde{G} - 1)$. Since $\tilde{E}$ and $\tilde{G}$ are smooth functions of (u, v), they have Taylor expansions $\tilde{E} = \sum_{i+j\leq 2} e_{ij}u^iv^j + o(r^2)$, $\tilde{G} = \sum_{i+j\leq 2} g_{ij}u^iv^j + o(r^2)$, where $o(r^k)$ denotes terms such that $o(r^k)/r^k \to 0$ as $r \to 0$. Equating coefficients on both sides of $u^2(\tilde{E} - 1) = v^2(\tilde{G} - 1)$ shows that all the e's and g's are zero except $e_{02} = g_{20} = k$, say. Then, $\tilde{E} = 1 + kv^2 + o(r^2)$, which implies that $G = r^2 + kr^4 + o(r^4)$. By Corollary 10.2.3(ii), $K = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial r^2}$. From the first part, $\sqrt{G} = r + \frac{1}{2}kr^3 + o(r^3)$, hence $K = -3k + o(1)$. Taking $r = 0$ gives $K(\mathbf{p}) = -3k$.

10.2.3 (i) $C_R = \int_0^{2\pi} \parallel \boldsymbol{\sigma}_\theta \parallel d\theta = \int_0^{2\pi} \sqrt{G}\, d\theta = \int_0^{2\pi} \left(R - \frac{1}{6}K(\mathbf{p})R^3 + o(R^3)\right) d\theta = 2\pi\left(R - \frac{1}{6}K(\mathbf{p})R^3 + o(R^3)\right)$.
(ii) Since $d\mathcal{A}_{\boldsymbol{\sigma}} = \sqrt{G}\, drd\theta$, the area $A_R = \int_0^R \int_0^{2\pi} \sqrt{G}\, drd\theta$ is equal to

$$2\pi \int_0^R \left(r - \frac{1}{6}K(\mathbf{p})r^3 + o(r^3)\right) dr = \pi R^2\left(1 - \frac{K(\mathbf{p})}{12}R^2 + o(R^2)\right).$$

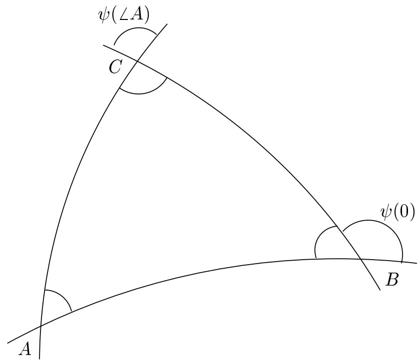
If $\mathcal{S} = S^2$, Exercise 6.5.3 gives $C_R = 2\pi \sin R = 2\pi(R = \frac{1}{6}R^3 + o(R^3))$, $A_R = 2\pi(1 - \cos R) = 2\pi(\frac{1}{2}R^2 - \frac{1}{24}R^4 + o(R^4))$. Since $K = 1$ these formulas agree with those in (i) and (ii).

10.2.4 (i) Let s be the arc-length of $\boldsymbol{\gamma}$, so that $ds/d\theta = \lambda$, and denote d/ds by a dot. The first of the geodesic equations (9.2) applied to $\boldsymbol{\gamma}$ gives $\ddot{r} = \frac{1}{2}G_r\dot{\theta}^2$. Since $r = f(\theta)$, this gives $\frac{1}{\lambda}\left(\frac{1}{\lambda}f'\right)' = \frac{1}{2\lambda^2}G_r$. This simplifies to give the stated equation.
(ii) Since $\boldsymbol{\sigma}_r$ and $\dot{\boldsymbol{\gamma}}$ are unit vectors, $\cos\psi = \boldsymbol{\sigma}_r.\dot{\boldsymbol{\gamma}} = \frac{1}{\lambda}\boldsymbol{\sigma}_r.(f'\boldsymbol{\sigma}_r + \boldsymbol{\sigma}_\theta) = f'/\lambda$. Also, $\boldsymbol{\sigma}_r \times \dot{\boldsymbol{\gamma}} = \frac{1}{\lambda}(\boldsymbol{\sigma}_r \times \boldsymbol{\sigma}_\theta) = \frac{\sqrt{G}}{\lambda}\mathbf{N}$, so $\sin\psi = \sqrt{G}/\lambda$. Hence, $\left(\frac{f'}{\lambda}\right)' = -\psi'\sin\psi = -\frac{\sqrt{G}}{\lambda}\psi'$, and so $\psi' = -\frac{1}{\sqrt{G}}\left(f'' - \frac{f'\lambda'}{\lambda}\right) = -\frac{1}{2\sqrt{G}}\frac{\partial G}{\partial r} = -\frac{\partial\sqrt{G}}{\partial r}$.

(iii) Using the formula for K in Corollary 10.2.3(ii) and the expression for the first fundamental form of $\boldsymbol{\sigma}$ in Exercise 9.5.1, we get

$$\begin{aligned}\iint_{\mathcal{T}} K d\mathcal{A}_{\boldsymbol{\sigma}} &= \int_0^\alpha \int_0^{f(\theta)} -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial r^2}\sqrt{G}\, dr d\theta \\ &= -\int_0^\alpha \left.\frac{\partial\sqrt{G}}{\partial r}\right|_{r=0}^{r=f(\theta)} d\theta = \int_0^\alpha \left(\psi' + \left.\frac{\partial\sqrt{G}}{\partial r}\right|_{r=0}\right) d\theta.\end{aligned}$$

By Exercise 10.2.2, $\sqrt{G} = r + o(r)$ so $\partial\sqrt{G}/\partial r = 1$ at $r = 0$, where $o(r)/r \to 0$ as $r \to 0$. Hence, $\iint_{ABC} K d\mathcal{A}_{\boldsymbol{\sigma}} = \psi(\alpha) - \psi(0) + \alpha = \gamma - (\pi - \beta) + \alpha = \alpha + \beta + \gamma - \pi$.



10.2.5 With the notation of Example 4.5.3 we have, on the median circle $t = 0$,

$$\boldsymbol{\sigma}_t = \left(-\sin\frac{\theta}{2}\cos\theta, -\sin\frac{\theta}{2}\sin\theta, \cos\frac{\theta}{2}\right), \quad \boldsymbol{\sigma}_\theta = (-\sin\theta, \cos\theta, 0),$$

hence $E = 1, F = 0, G = 1$ and $\mathbf{N} = \left(-\cos\frac{\theta}{2}\cos\theta, -\sin\frac{\theta}{2}\sin\theta, -\sin\frac{\theta}{2}\right)$;

$$\boldsymbol{\sigma}_{tt} = \mathbf{0}, \;\; \boldsymbol{\sigma}_{t\theta} = \left(-\frac{1}{2}\cos\frac{\theta}{2}\cos\theta + \sin\frac{\theta}{2}\sin\theta, -\frac{1}{2}\cos\frac{\theta}{2}\sin\theta - \sin\frac{\theta}{2}\cos\theta, -\frac{1}{2}\sin\frac{\theta}{2}\right),$$

giving $L = 0, M = \frac{1}{2}$. Hence, $K = (LN - M^2)/(EG - F^2) = -1/4$. Since $K \neq 0$, the Theorema Egregium implies that the Möbius band is not locally isometric to a plane.

10.2.6 The catenoid has first fundamental form $\cosh^2 u(du^2 + dv^2)$ and its Gaussian curvature is $K = -\text{sech}^4 u$ (Exercise 8.1.2). If f is an isometry of the catenoid, let $f(\boldsymbol{\sigma}(u, v)) = \boldsymbol{\sigma}(\tilde{u}, \tilde{v})$. By the Theorema Egregium, $\text{sech}^4 u = \text{sech}^4 \tilde{u}$, so $\tilde{u} = \pm u$; reflecting in the plane $z = 0$ changes u to $-u$, so assume that the sign is $+$. Let $\tilde{v} = f(u, v)$; the first fundamental form of $\boldsymbol{\sigma}(u, f(u, v))$ is

$$(\cosh^2 u + f_u^2)du^2 + 2f_u f_v du dv + f_v^2 \cosh^2 u dv^2.$$

Hence, $\cosh^2 u = \cosh^2 u + f_u^2$, $f_u f_v = 0$ and $f_v^2 \cosh^2 u = \cosh^2 u$. So $f_u = 0$, $f_v = \pm 1$ and $f = \pm v + \alpha$, where α is a constant. If the sign is $+$ we have a rotation by α about the z-axis; if the sign is $-$ we have a reflection in the plane containing the z-axis and making an angle $\alpha/2$ with the xz-plane.

10.2.7 Using Corollary 10.2.3(i), we find that

$$K = \frac{1}{4}\left(\frac{m(2-m)}{u^n v^2} + \frac{n(2-n)}{u^2 v^m}\right).$$

Hence, the surface is flat if and only if $(m, n) = (0, 0)$, $(2, 0)$, $(0, 2)$ or $(2, 2)$.

If $m = n = 0$ the first fundamental form is $du^2 + dv^2$ so the surface is locally isometric to a plane (Corollary 6.2.3). If $(m, n) = (2, 0)$, we get $v^2du^2 + dv^2$, which is the first fundamental form pf a cone (Example 6.1.5), and this is locally isometric to a plane by Exercise 6.2.5. Similarly if $(m, n) = (0, 2)$. The case in which $m = n = 2$ is less obvious, but the reparametrization $U = 2^{-1/2}uv$, $V = \ln \frac{u}{v}$ transforms the first fundamental form $v^2du^2 + u^2dv^2$ into $dU^2 + U^2dV^2$, which is the first fundamental form of a cone.

10.2.8 Using the Christoffel symbols calculated in Exercise 7.4.4, the second of the Gauss equations (Proposition 10.1.2) gives

$$K \cos\theta = 0 - (\theta_u \cot\theta)_v + 0 - (-\theta_u \operatorname{cosec}\theta)(-\theta_v \operatorname{cosec}\theta) = -\theta_{uv}\cot\theta,$$

hence the result.

10.2.9 By Exercise 6.2.5 and the Theorema Egregium, $K = 0$ for a generalized cylinder and a generalized cone. But $K \neq 0$ for a sphere so by the Theorema Egregium again, there can be no local isometry between a sphere and a generalized cylinder or cone.

10.2.10 The first fundamental forms of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ are

$$\left(1 + \frac{1}{u^2}\right) du^2 + u^2 dv^2 \quad \text{and} \quad du^2 + (u^2+1)dv^2,$$

respectively. Since these are different, the map $\boldsymbol{\sigma}(u, v) \mapsto \tilde{\boldsymbol{\sigma}}(u, v)$ is not an isometry. Nevertheless, by Corollary 10.2.3(i), the Gaussian curvatures are equal:

$$K = \tilde{K} = -\frac{1}{2\sqrt{u^2+1}}\frac{\partial}{\partial u}\left(\frac{2u}{\sqrt{1+u^2}}\right) = -\frac{1}{(1+u^2)^2}.$$

If $\boldsymbol{\sigma}(u, v) \mapsto \tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ is an isometry, the Theorema Egregium tells us that $-\frac{1}{(1+u^2)^2} = -\frac{1}{(1+\tilde{u}^2)^2}$, so $\tilde{u} = \pm u$; let $\tilde{v} = f(u, v)$. The first fundamental form of $\tilde{\boldsymbol{\sigma}}(\pm u, f(u, v))$ is $(1 + (1+u^2)f_u^2)du^2 + 2(1+u^2)f_u f_v du dv + (1+u^2)f_v^2 dv^2$; this

is equal to the first fundamental form of $\boldsymbol{\sigma}(u,v) \Longleftrightarrow 1+(1+u^2)f_u^2 = 1+1/u^2$, $f_u f_v = 0$ and $(1+u^2)f_v^2 = u^2$. The middle equation gives $f_u = 0$ or $f_v = 0$, but these are both impossible by the other two equations. Hence, the isometry does not exist.

10.2.11 From the solution of Exercise 8.1.9, the Gaussian curvature of the torus (in the parametrization of Exercise 4.2.5) is

$$K = \frac{\cos\theta}{b(a+b\cos\theta)} = \frac{1}{b^2} - \frac{a}{b^2(a+b\cos\theta)}.$$

If $\boldsymbol{\sigma}(\theta,\varphi) \mapsto \boldsymbol{\sigma}(\tilde\theta,\tilde\varphi)$ is an isometry of the torus, the Theorema Egregium implies that we must have $\cos\theta = \cos\tilde\theta$, and hence $\tilde\theta = \pm\theta$ (up to adding integer multiples of 2π). Since the first fundamental form is $b^2d\theta^2 + (a+b\cos\theta)^2d\varphi^2$, we must therefore have $d\tilde\varphi^2 = d\varphi^2$, and hence $\partial\tilde\varphi/\partial\theta = 0$, $\partial\tilde\varphi/\partial\varphi = \pm 1$. Hence, $\tilde\varphi = \pm\varphi +$ constant.

10.2.12 Since all parameter curves are pre-geodesics we have, by Exercise 9.2.8,

$$EE_v + FE_u = 2EF_u, \quad GG_u + FG_v = 2GF_v.$$

If the parameter curves intersect orthogonally, $F = 0$ so these equations give $E_v = G_u = 0$, so the first fundamental form is $E(u)du^2 + G(v)dv^2$. Reparametrizing by $U = \int\sqrt{E(u)}du$, $V = \int\sqrt{G(v)}dv$, the first fundamental form becomes $dU^2 + dV^2$. Hence, the surface is locally isometric to the plane, and so is flat by the Theorema Egregium.

The result is not true without the assumption of orthogonality. Example 10.4.3 gives a parametrization $\boldsymbol{\sigma}(u,v)$ of S^2 (actually of a hemisphere) with the property that the pre-geodesics on S^2 correspond exactly to the straight lines in the uv-plane. In particular, all the parameter curves are pre-geodesics.

10.3.1 Arguing as in the proof of Theorem 10.3.4, we suppose that J attains its maximum value > 0 at some point $\mathbf{p} \in \mathcal{S}$ contained in a patch $\boldsymbol{\sigma}$ of $\mathcal{S}$. We can assume that the principal curvatures κ_1 and κ_2 of $\boldsymbol{\sigma}$ satisfy $\kappa_1 > \kappa_2 > 0$ everywhere. Since $H = \frac{1}{2}(\kappa_1+\kappa_2)$, $\kappa_1 > H$ and $J = 4(\kappa_1 - H)^2$. Thus, J increases with κ_1 when $\kappa_1 > H$, so κ_1 must have a maximum at $\mathbf{p}$, and then $\kappa_2 = 2H - \kappa_1$ has a minimum there. By Lemma 10.3.5, $K \leq 0$ at $\mathbf{p}$, contradicting the assumption that $K > 0$ everywhere.

10.3.2 We start with the parametrization $\boldsymbol{\sigma}(U,V) = (f(U)\cos V, f(U)\sin V, g(U))$, where $f(U) = e^U$, $g(U) = \int\sqrt{1-e^{2U}}dU$. The first and second fundamental forms are $dU^2 + e^{2U}dV^2$ and $\frac{-e^U}{\sqrt{1-e^{2U}}}dU^2 + e^U\sqrt{1-e^{2U}}dV^2$, respectively. In the notation of the proof of Proposition 10.3.2, $\kappa_1 = -1/e^U\sqrt{1-e^{2U}}$, $\kappa_2 = e^{-U}\sqrt{1-e^{2U}}$. So we are in case (ii) of the proof and $\tan\omega = \sqrt{e^{-2U}-1}$. We find that $e(U) =$

$E/\sin^2\omega = \frac{1}{1-e^{2U}}$, $g(V) = G\sec^2\omega = 1$. So $\tilde{V} = V$ and $\tilde{U} = \int \frac{dU}{\sqrt{1-e^{2U}}} = -\cosh^{-1}(e^{-U}) - c$ for some constant c. So $U = -\ln(\cosh(\tilde{U}+c))$. Hence, $\theta = 2\omega = 2\tan^{-1}\sqrt{e^{-2U}-1} = 2\tan^{-1}\sqrt{\cosh^2(\tilde{U}+c)-1} = 2\tan^{-1}\sinh(\tilde{U}+c)$. Finally, $u = \frac{1}{2}(\tilde{U}+\tilde{V}), v = \frac{1}{2}(\tilde{V}-\tilde{U})$, so $\tilde{U} = u - v$.

10.3.3 We have $E = G = 1$, $F = \cos\theta$ and $\theta_{uv} = \sin\theta$. Hence, $\sqrt{EG-F^2} = \theta_{uv}$. Consider the quadrilateral bounded by the parameter curves $u = u_0$, $u = u_1$, $v = v_0$ and $v = v_1$, and let

α_1 be the angle between the parameter curves $u = u_0$ and $v = v_0$,
α_2 be the angle between the parameter curves $u = u_0$ and $v = v_1$,
α_3 be the angle between the parameter curves $u = u_1$ and $v = v_0$,
α_4 be the angle between the parameter curves $u = u_1$ and $v = v_1$.

The area of the quadrilateral is

$$\begin{aligned}\int_{v_0}^{v_1}\int_{u_0}^{u_1}\theta_{uv}dudv &= \int_{v_0}^{v_1}(\theta_v(u_1,v)-\theta_v(u_0,v))dv\\ &= \theta(u_1,v_1)-\theta(u_1,v_0)-\theta(u_0,v_1)+\theta(u_0,v_0)\\ &= \alpha_4-(\pi-\alpha_3)-(\pi-\alpha_2)+\alpha_1\\ &= \alpha_1+\alpha_2+\alpha_3+\alpha_4-2\pi.\end{aligned}$$

10.4.1 Let $f : \mathcal{S} \to \tilde{\mathcal{S}}$ be a local diffeomorphism that takes unit-speed geodesics to unit-speed geodesics. Let $\mathbf{p} \in \mathcal{S}$ and let $\mathbf{0} \neq \mathbf{v} \in T_{\mathbf{p}}\mathcal{S}$. There is a unique geodesic $\boldsymbol{\gamma}(t)$ on $\mathcal{S}$ such that $\boldsymbol{\gamma}(0) = \mathbf{p}$ and $\dot{\boldsymbol{\gamma}}(0) = \mathbf{v}/\parallel \mathbf{v} \parallel$. Then $\boldsymbol{\gamma}$ is unit-speed so $\tilde{\boldsymbol{\gamma}} = f \circ \boldsymbol{\gamma}$ is a unit-speed geodesic. In particular, $\dot{\tilde{\boldsymbol{\gamma}}} = D_{\mathbf{p}}f(\mathbf{v}/\parallel \mathbf{v} \parallel)$ is a unit vector, i.e. $\parallel D_{\mathbf{p}}f(\mathbf{v}) \parallel = \parallel \mathbf{v} \parallel$. This means that $D_{\mathbf{p}}f : T_{\mathbf{p}}\mathcal{S} \to T_{f(\mathbf{p})}\tilde{\mathcal{S}}$ is an isometry, so f is a local isometry.

10.4.2 Local isometries take geodesics to geodesics by Corollary 9.2.7. If we apply a dilation $\mathbf{v} \mapsto a\mathbf{v}$, where $a \neq 0$ is a constant, to a surface, the first fundamental form is multiplied by a^2 so the Christoffel symbols are unchanged (see Proposition 7.4.4). By Proposition 9.2.3, the geodesic equations are unchanged. It follows that dilations take geodesics to geodesics. Hence, any composite of local isometries and dilations also takes geodesics to geodesics.

The converse is false: the map from the xy-plane to itself given by $(x, y) \mapsto (x, 2y)$ takes geodesics to geodesics (as it takes straight lines to straight lines) but is not the composite of a dilation and a local isometry.

10.4.3 (i) This is true because F is conformal.

(ii) The parameter curve $u \mapsto \boldsymbol{\sigma}(u, v_0)$ is a geodesic on $\boldsymbol{\sigma}$ for any fixed v_0 by construction of the geodesic patch $\boldsymbol{\sigma}$. Since F is a geodesic local diffeomorphism, $u \mapsto F(\boldsymbol{\sigma}(u, v_0))$ is a pre-geodesic on $\tilde{\boldsymbol{\sigma}}$. Hence, for some smooth function $u(t)$, $t \mapsto F(\boldsymbol{\sigma}(u(t), v_0))$ is a geodesic on $\tilde{\boldsymbol{\sigma}}$. The second geodesic equation in Theorem 9.2.1 gives $\lambda_v \dot{u}^2 = 0$, so $\lambda_v = 0$ and λ is independent of v.
(iii) Let $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$; we can assume that $\boldsymbol{\gamma}$ is unit-speed. Since the first fundamental form of $\boldsymbol{\sigma}$ is $du^2 + Gdv^2$, the parameter curves $v =$ constant and $u =$ constant intersect orthogonally and unit vectors parallel to them are $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v/\sqrt{G}$, respectively. If the oriented angle between $\dot{\boldsymbol{\gamma}}$ and the curve $v =$ constant is θ, we have $\dot{\boldsymbol{\gamma}} = \cos\theta \boldsymbol{\sigma}_u + G^{-1/2} \sin\theta \boldsymbol{\sigma}_v = \dot{u}\boldsymbol{\sigma}_u + \dot{v}\boldsymbol{\sigma}_v$. Hence, $\dot{u} = \cos\theta$ and $\dot{v} = \frac{\sin\theta}{\sqrt{G}}$. The first geodesic equation in Theorem 9.2.1 gives $\ddot{u} = \frac{1}{2} G_u \dot{v}^2$, i.e. $\dot{\theta}\sin\theta = \frac{1}{2} G_u \dot{v}^2$. Substituting for $\dot{v}$ gives $\frac{d\theta}{dv} = \frac{\dot{\theta}}{\dot{v}} = \frac{G_u \dot{v}}{2\sin\theta} = G_u/2\sqrt{G}$.
(iv) Apply (iii) to $F \circ \boldsymbol{\gamma}$ and use the fact that F is conformal.
(v) Parts (iii) and (iv) imply $(\lambda G)_u = \lambda G_u$, hence $\lambda_u G = 0$, hence $\lambda_u = 0$, i.e. λ is independent of u. By (ii), λ is constant.
(vi) If $D_{\lambda^{-1/2}}$ is the dilation by a factor $\lambda^{-1/2}$, the composite $D_{\lambda^{-1/2}} \circ F$ preserves the first fundamental form and so is a local isometry, say $\mathcal{G}$. Then, $F = D_{\lambda^{1/2}} \circ \mathcal{G}$.

Chapter 11

11.1.1 Let l meet the real axis at b and suppose that $\mathfrak{Re}(a) > b$ (the case $\mathfrak{Re}(a) < b$ is similar). The semicircle with centre c on the real axis and radius $|a - d|$ passes through a and does not meet l provided that $|a - d| \leq |d - b|$, i.e. provided $d \geq (|a|^2 - b^2)/(\mathfrak{Re}(a) - b)$.

11.1.2 Suppose that a and b lie on a half-line geodesic, say $a = r + is, b = r + it$, where $r, s, t \in \mathbb{R}$ and $t > s$. Then, $d_{\mathcal{H}}(a, b) = \int_s^t \frac{dw}{w} = \ln(t/s) = d$, say, so $t/s = e^d$. On the other hand, the formula in Proposition 11.1.4 gives $2\tanh^{-1}\left(\frac{t-s}{t+s}\right) = 2\tanh^{-1}\left(\frac{e^d-1}{e^d+1}\right) = 2\tanh^{-1}(\tanh\frac{d}{2}) = d$.

11.1.3 The required hyperbolic line cannot be a half-line, so must be a semicircle with centre the origin, and it must have radius $|a|$.

11.1.4 By Proposition 11.1.4, $z \in \mathcal{C}_{a,R} \iff 2\tanh^{-1}\left|\frac{z-a}{z-\bar{a}}\right| = R$. If $\lambda = \tanh(R/2)$, this is equivalent to $(1 - \lambda^2)|z|^2 - (\bar{a} - \lambda^2 a)z - (a - \lambda^2\bar{a})\bar{z} + (1 - \lambda^2)|a|^2 = 0$. According to Proposition A.2.3, this is the equation of a circle provided that $\lambda^2 < 1$, which is obvious, and $|\bar{a} - \lambda^2 a|^2 > (1 - \lambda^2)^2|a|^2$. This condition reduces to $2|a|^2 > a^2 + \bar{a}^2$. Writing $a = |a|e^{i\theta}$ this becomes $\cos 2\theta < 1$, which is true because $a \in \mathcal{H}$ implies $0 < \theta < \pi$.
$\mathcal{C}_{ic,R}$ will be a circle with centre on the imaginary axis, say at ib. Then $\mathcal{C}_{a,R}$ intersects the imaginary axis at the points $i(b \pm r)$, so these two points must

be a hyperbolic distance $2R$ apart, i.e. $2R = 2\tanh^{-1}\left|\frac{2ir}{2ib}\right|$. This gives $r = b\tanh R$, which is equivalent to $R = \frac{1}{2}\ln\frac{b+r}{b-r}$. Next, the points $i(b \pm r)$ are the same hyperbolic distance from ic, so $\left|\frac{i(b+r)-ic}{i(b+r)+ic}\right| = \left|\frac{i(b-r)-ic}{i(b-r)+ic}\right|$. This gives $(b+r-c)(b-r+c) = (c+r-b)(c+r+b)$, which simplifies to $c^2 = b^2 - r^2$. Parametrizing $\mathcal{C}_{ic,R}$ by $v = r\cos\theta, w = b + r\sin\theta$, and denoting $d/d\theta$ by a dot, the circumference of $\mathcal{C}_{ic,R}$ is $\int_0^{2\pi}\frac{\sqrt{\dot{v}^2+\dot{w}^2}}{w^2}d\theta = \int_0^{2\pi}\frac{r}{b+r\sin\theta}d\theta = \frac{2\pi r}{\sqrt{b^2-r^2}} = 2\pi\sinh R$. The area inside $\mathcal{C}_{ic,R}$ is $\iint_{\mathrm{int}(\mathcal{C}_{ic,R})}\frac{dvdw}{w} = \int_{\mathcal{C}_{ic,R}}\frac{dv}{w}$ by Green's theorem, which $= \int_0^{2\pi}\frac{-r\sin\theta}{b+r\sin\theta}d\theta = \frac{2\pi b}{\sqrt{b^2-r^2}} - 2\pi = 2\pi(\cosh R - 1)$.

The circumference is $2\pi\sinh R = 2\pi(R + \frac{1}{6}R^3 + o(R^3))$ and the area is $2\pi(\cosh R - 1) = 2\pi(\frac{1}{2}R^2 + \frac{1}{24}R^4 + o(R^4)))$. Since $K = -1$ these formulas are consitent with those in Exercise 10.2.7. (See Exercise 10.2.4 for the $o(\ )$ notation.)

11.1.5 Suppose that l is the imaginary axis for simplicity. If $a = re^{i\theta}$, with $0 < \theta < \pi/2$, the distance of a from l is the distance of a from the point ir on l. From the proof of Proposition 11.1.4, this distance is $-\ln\tan\frac{\theta}{2}$. Hence, the points at a fixed distance from l are those with a constant value of θ, i.e. the straight half-lines passing through the origin. Of course, there are two such half-lines for any given distance, each being the (Euclidean) reflection of the other in l. These lines are not geodesics (unless the distance is zero, in which case they coincide with l), as they are not perpendicular to the real axis.

11.1.6 The region $\{(u,v)\,|\,u < 0, -\pi < v < \pi\}$ in the parametrization of the pseudosphere corresponds to the region $\{(v,w)\,|\,-\pi < v < \pi, w > 1\}$ in $\mathcal{H}$, a semi-infinite rectangle in the upper half of the vw-plane.

11.2.1 By Proposition 11.2.3, there is an isometry that takes a to i and b to ir, say, where $r > 0$. Since isometries leave distances unchanged, we need only prove the result for $a = i, b = ir$. Assume that $r > 1$ (the case $r < 1$ is similar). Then, $d_{\mathcal{H}}(a,b) = \ln r$. On the other hand, if $\boldsymbol{\gamma}(t) = v(t) + iw(t)$ is any curve in $\mathcal{H}$ with $\boldsymbol{\gamma}(t_0) = i, \boldsymbol{\gamma}(t_1) = ir$, say, the length of the part of $\boldsymbol{\gamma}$ between a and b is $\int_{t_0}^{t_1}\frac{\sqrt{\dot{v}^2+\dot{w}^2}}{w}dt \geq \int_{t_0}^{t_1}\frac{\dot{w}}{w}dt = \int_1^r\frac{dw}{w} = \ln r$.

11.2.2 By applying an isometry we can assume that l is a half-line, and then the result was proved in Exercise 11.1.1.

11.2.3 By applying an isometry, we can assume that l is the imaginary axis. Then, m must be the semicircle with centre the origin and radius $|a|$. Let $a = \rho e^{i\theta}$, where $\rho > 0, -\pi < \theta < \pi$, and let $c = it$, $t > 0$. Since $\tanh^{-1}x$ is a strictly increasing function of x, we have to show that $\left|\frac{a-it}{a+it}\right| > \left|\frac{a-i\rho}{a+i\rho}\right|$ if $t \neq \rho$. The second expression equals $\frac{1-\sin\theta}{1+\sin\theta}$, and the difference is $\frac{2(\rho-t)^2}{|a+it|^2}\frac{\sin\theta}{1+\sin\theta}$, which is > 0 if $t \neq \rho$.

11.2.4 (i) If $a \in \mathcal{H}$, let l' and m' be the unique hyperbolic lines passing through a and perpendicular to l and m, respectively (Exercise 11.2.3). Let b and c be

the intersections of l' and m' with l and m, respectively. We are given that $F(b) = b$ and $F(c) = c$, so $F(l') = l'$ and $F(m') = m'$ as l' and m' are the unique hyperbolic lines passing through b and c and perpendicular to l and m. Since a is the unique point of intersection of l' and m', we must have $F(a) = a$.
(ii) Either F or $\mathcal{I}_{0,1} \circ F$ fixes l, m and the interior of the semicircle m. Next, either F, $\mathcal{I}_{0,1} \circ F$, $\mathcal{R}_0 \circ F$ or $\mathcal{R}_0 \circ \mathcal{I}_{0,1} \circ F$ fixes l, m, the interior of the semicircle m and the region $\mathcal{H}_{>0} = \{z \in \mathcal{H} \,|\, \mathfrak{Re}(z) > 0\}$ to the right of l. Let G be this isometry. Then, G fixes each point of m because there is a unique point of m at any given distance $d > 0$ from i in the region $\mathcal{H}_{>0}$. Similarly, G fixes each point of l. Hence, G is the identity by (i).
(iii) Let F be any isometry of $\mathcal{H}$. By the proof of Proposition 11.2.3, there is an isometry G that is a composite of elementary isometries and which takes $F(i)$ to i and $F(l)$ to l. Then, $G \circ F$ is an isometry that fixes l and i. As m is the unique hyperbolic line intersecting l perpendicularly at i, $G \circ F$ fixes m. By (iii), $G \circ F$ is one of four composites of elementary isometries. It follows that F is a composite of elementary isometries.
(iv) By (iii) it suffices to prove that every elementary isometry is a composite of reflections and inversions in lines and circles perpendicular to the real axis. For reflections and inversions there is nothing to prove, so we need only consider translations and dilations. But, if $a \in \mathbb{R}$, $\mathcal{T}_a = \mathcal{R}_0 \circ \mathcal{R}_1$, where $\mathcal{R}_1$ is the reflection in the line $\mathfrak{Re}(z) = a/2$; and if $a > 0$, $\mathcal{D}_a = \mathcal{I}_0 \circ \mathcal{I}$, where $\mathcal{I}$ is inversion in the circle with centre the origin and radius $\sqrt{a}$.

11.2.5 (i) This is obvious from the proof of Proposition A.1.2(ii).
(ii) If $a, b, c, d \in \mathbb{R}$, a calculation shows that $\mathfrak{Im}(M(z)) = \frac{ad-bc}{|cz+d|^2}\mathfrak{Im}(z)$. If $ad - bc > 0$ this is > 0 whenever $\mathfrak{Im}(z) > 0$. Conversely, suppose that M takes $\mathcal{H}$ to itself. Assume that $c \neq 0$ and $d \neq 0$ (the cases in which $c = 0$ or $d = 0$ are similar but easier). Then, M must take the real axis to itself (as it must take the lower half-plane $-\mathcal{H}$ to itself), i.e. $\frac{az+b}{cz+d} \in \mathbb{R}$ if $z \in \mathbb{R}$. Taking $z = 0$ gives $b/d = \lambda \in \mathbb{R}$, say. Letting $z \to \infty$ gives $a/c = \mu \in \mathbb{R}$, say. Then, $\frac{az+b}{cz+d} = \mu + \frac{\lambda-\mu}{\frac{c}{d}z+1}$. This is real whenever z is real, so $c/d = \nu \in \mathbb{R}$, say. Hence, a, b, c, d are, up to an overall multiple, equal to the real numbers $\mu\nu, \lambda, \nu, 1$, respectively. The condition $ad - bc > 0$ now follows from the previous calculation.
(iii) Following the proof (and the notation) of Proposition A.2.2, we have $M = \mathcal{T}_{b/d} \circ \mathcal{D}_{a/d}$ if $c = 0$ (and then $d \neq 0$), while if $c \neq 0$,

$$M = \mathcal{T}_{a/c} \circ \mathcal{D}_{(ad-bc)/c^2} \circ (-K) \circ \mathcal{T}_{d/c}.$$

Hence, it suffices to show that $-K$ is a composite of elementary isometries. But $-K = \mathcal{R}_0 \circ \mathcal{I}_{0,1}$ in the notation of Exercise 11.2.4.
(iv) J is reflection in the imaginary axis and hence an isometry of $\mathcal{H}$, so the result follows from (i).

(v) This follows from the fact that, if M is a real Möbius transformation, so is $J \circ M \circ J$. For example, if M_1, M_2 are real Möbius transformations, then

$$(M_1 \circ J) \circ (M_2 \circ J) = M_1 \circ (J \circ M_2 \circ J)$$

is a composite of real Möbius transformations, hence a real Möbius tranformation by (i).
(vi) By (v) and Exercise 11.2.4(iii), it suffices to prove that every elementary isometry of $\mathcal{H}$ is a Möbius isometry. If $a \in \mathbb{R}$, $\mathcal{T}_a$ is real Möbius and $\mathcal{R}_a = \mathcal{T}_{2a} \circ J$. If $a > 0$ then $\mathcal{D}_a$ is real Möbius. Finally, if $a \in \mathbb{R}, r > 0$ then

$$\mathcal{I}_{a,r} = \mathcal{T}_a \circ \mathcal{D}_{r^2} \circ \mathcal{I}_{0,1} \circ \mathcal{T}_{-a}$$

(see the proof of Proposition 11.2.1), so it suffices to prove that $\mathcal{I}_{0,1}$ is a Möbius isometry; but $\mathcal{I}_{0,1} = (-K) \circ J$, where $-K(z) = -1/z$ is real Möbius.

11.2.6 By Proposition 11.1.3(i), there is a unique hyperbolic line l, say, passing through a and b. By Proposition 11.2.3, by applying a suitable isometry of $\mathcal{H}$ we can assume that l is the imaginary axis. Let $\mathcal{C}$ and $\mathcal{C}'$ be the hyperbolic circles with centres a and b, respectively, which pass through c (Exercise 11.1.4). These are Euclidean circles with centres on the imaginary axis.

Suppose for a contradiction that the stated inequality is false, and let $a = iw$, $b = iw'$, where $w' > w$ (interchanging the roles of a, b if necessary). Then, $\mathcal{C}$ intersects the imaginary axis at a point ir, say, with $r > w$; we must have $r < w'$ since $r \geq w'$ implies that $d(a,b) \leq d(a, ir) = d(a,c)$, which contradicts our assumption. Similarly, $\mathcal{C}'$ intersects the imaginary axis at a point is, say, between a and b. Since $\mathcal{C}$ and $\mathcal{C}'$ intersect, we must have $s \leq r$; in fact, $s < r$ for if $s = r$ then $\mathcal{C}$ and $\mathcal{C}'$ touch, contradicting the fact that a, b, c do not lie on a geodesic. Then,

$$d(a,c) + d(c,b) = d(a, ir) + d(is, b) > d(a, ir) + d(ir, b) = d(a,b),$$

a contradiction.

11.2.7 Let F be an isometry of $\mathcal{H}$ that takes l to the imaginary axis. By Exercise 11.1.5, the equidistant curves are then a pair of Euclidean half-lines l_1, l_2 passing through the origin. The isometry F^{-1} of $\mathcal{H}$, being a composite of reflections and inversions (Exercise 11.2.4), takes Euclidean lines and circles to Euclidean lines and circles (Appendix 2). Since the real axis is the only Euclidean straight line passing through a and b, F^{-1} must take l_1, l_2 to a pair of circular arcs passing through a and b. We observed in Exercise 11.1.5 that these are not geodesics.

11.2.8 By applying a suitable isometry, we can assume that a and d are on the imaginary axis. The centre of the semicircle geodesic passing through a and b is $(R, 0)$,

where $a = iR\cot\frac{1}{2}\alpha$, and its radius is $R\operatorname{cosec}\frac{1}{2}\alpha$. Let $d = it$. We must have $R\operatorname{cosec}\frac{1}{2}\alpha < R + t$. Then,

$$d(a,d) = \ln\left(\frac{R}{t}\cot\frac{1}{2}\alpha\right) < \ln\left(\frac{\cot\frac{1}{2}\alpha}{\operatorname{cosec}\frac{1}{2}\alpha - 1}\right) = f(\alpha).$$

11.2.9 As in the proof of Theorem 11.2.4, we can assume that $a = a'$, that a, c and c' are all on the imaginary axis, that b and b' are on the same side of the imaginary axis, and that c and c' are either both 'above' or both 'below' a on the imaginary axis. Then, clearly, $c = c'$. Since the angles of the two triangles at a are equal, the geodesic through a and b is the same as that through a and b'. Since $d(a,b) = d(a,b')$ and since b and b' are on the same side of the imaginary axis, we must have $b = b'$.

11.2.10 The proof is the same as in Euclidean geometry. Let $a, b \in \mathcal{H}$, let d be the mid-point of the geodesic l passing through a and b, and let m be the geodesic passing through d perpendicular to l. If c is any point on m then, by the preceding exercise, the triangles with vertices a, d, c and b, d, c are congruent. In particular, $d(a,c) = d(b,c)$. Hence, m is the set of points equidistant from a and b.

11.3.1 The distance we want is $2\tanh^{-1}\left|\frac{\mathcal{P}^{-1}(b)-\mathcal{P}^{-1}(a)}{\mathcal{P}^{-1}(b)-\overline{\mathcal{P}^{-1}(a)}}\right|$. Now use $\mathcal{P}^{-1}(z) = \frac{z+1}{i(z-1)}$. The algebra is straightforward.

11.3.2 By Proposition 11.2.3, there is an isometry F that takes l to the real axis and the point of intersection of l and m to the origin. Then F must take m to the imaginary axis as this is the unique hyperbolic line through the origin perpendicular to the real axis. The number of such isometries is the number of isometries that take the real axis to itself and the imaginary axis to itself. If G is such an isometry, then either G, $\mathcal{R}_0 \circ G$, $\mathcal{R}_1 \circ G$ or $\mathcal{R}_1 \circ \mathcal{R}_0 \circ G$ fix the real and imaginary axes and also each quadrant into which the disc $\mathcal{D}_P$ is divided by the axes. If H is this isometry, the argument used in the solution of Exercise 11.2.4(ii) shows that H must be the identity map. Hence, $G = \mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_0 \circ \mathcal{R}_1$ or the identity map.

11.3.3 Let us call a Möbius transformation of the type in the statement of the exercise a *hyperbolic* Möbius transformation. Since $\mathcal{P}$ is a Möbius transformation, the Möbius transformations that take $\mathcal{D}_P$ to itself are those of the form $\mathcal{P}M\mathcal{P}^{-1}$, where M is a Möbius transformation that takes $\mathcal{H}$ to itself, i.e. a real Möbius transformation (Exercise 11.2.5). If $M(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$, we find that $\mathcal{P}M\mathcal{P}^{-1}(z) = \frac{(a+d+i(b-c))z+a-d-i(b+c)}{(a-d+i(b+c))z+a+d-i(b-c)}$. Since

$$|a+d+i(b-c)|^2 - |a-d-i(b+c)|^2 = 4(ad-bc) > 0,$$

$\mathcal{P}M\mathcal{P}^{-1}$ is hyperbolic. Conversely, we have to show that if M is a hyperbolic Möbius transformation, then $\mathcal{P}^{-1}M\mathcal{P}$ is a real Möbius transformation. The calculation is similar to that already given.

11.3.4 By Exercise 11.2.5(iii), the isometries of $\mathcal{H}$ are of the form M or $M \circ J$ where M is real Möbius and $J(z) = -\bar{z}$. Hence, the isometries of $\mathcal{D}_P$ are $\mathcal{P}M\mathcal{P}^{-1}$ and $\mathcal{P}(M \circ J)\mathcal{P}^{-1} = \mathcal{P}M\mathcal{P}^{-1} \circ \mathcal{P}J\mathcal{P}^{-1}$. But $\mathcal{P}M\mathcal{P}^{-1}$ is hyperbolic and $\mathcal{P}J\mathcal{P}^{-1}(z) = \mathcal{P}J\left(\frac{z+1}{i(z-1)}\right) = \mathcal{P}\left(\frac{\bar{z}+1}{i(\bar{z}-1)}\right) = \mathcal{P}(\mathcal{P}^{-1}(\bar{z})) = \bar{z}$.

11.3.5 We know that every isometry of $\mathcal{H}$ is the composite of reflections $\mathcal{R}_a$ and inversions $\mathcal{I}_{a,r}$ with $a \in \mathbb{R}$, $r > 0$ (Exercise 11.2.4(iv)). Now, $\mathcal{I}_{a,r} = \mathcal{T}_a \circ \mathcal{D}_{r^2} \circ \mathcal{I}_{0,1} \circ \mathcal{T}_{-a}$ and any translation $\mathcal{T}_a$ ($a \in \mathbb{R}$) is the composite of reflections $\mathcal{R}_0 \circ \mathcal{R}_{a/2}$. It therefore suffices to show that, if F is any isometry of $\mathcal{H}$ of the form $\mathcal{R}_a$ ($a \in \mathbb{R}$), $\mathcal{D}_a$ ($a > 0$) or $\mathcal{I}_{0,1}$, then $\mathcal{P} \circ F \circ \mathcal{P}^{-1}$ is a composite of isometries of $\mathcal{D}_P$ of the types in Proposition 11.3.3. We find that (a) if $a \neq 0$, $\mathcal{P} \circ \mathcal{R}_a \circ \mathcal{P}^{-1} = \mathcal{I}_{b,r}$, where $b = \frac{1+ia}{ia}$, $r = 1/|a|$; (b) $\mathcal{P} \circ \mathcal{R}_0 \circ \mathcal{P}^{-1}$ is reflection in the real axis; (c) $\mathcal{P} \circ \mathcal{I}_{0,1} \circ \mathcal{P}^{-1}$ is reflection in the imaginary axis; (d) if $a > 0$, $\mathcal{P} \circ \mathcal{D}_a \circ \mathcal{P}^{-1}$ is the composite of two inversions of the type in Proposition 11.3.3(i), namely $\mathcal{I}_{c,\sqrt{c^2-1}} \circ \mathcal{I}_{b,\sqrt{b^2-1}}$, where b, c are real numbers such that $b^2 > 1$, $c^2 > 1$ and $a = f(c)/f(b)$ where $f(x) = \frac{x+1}{x-1}$. (We can take b to be any real number > 1 and distinct from $1/f(a)$, then choose $c = f(af(b))$; then, $f(c) = f(f(af(b))) = af(b)$ using the property $f(f(x)) = x$ for all $x \neq 1$.)

11.3.6 Suppose first that $\gamma = \pi/2$. The cosine rule gives $\cosh C = \cosh A \cosh B$ and $\cosh A = \cosh B \cosh C - \sinh B \sinh C \cos\alpha$. Eliminating $\cosh C$ gives $\cos\alpha = \cosh A \sinh B / \sinh C$. Hence,

$$\begin{aligned}
\sin^2\alpha \sinh^2 C &= \sinh^2 C - \cosh^2 A \sinh^2 B \\
&= \cosh^2 A \cosh^2 B - 1 - \cosh^2 A \sinh^2 B \\
&= \cosh^2 A - 1 = \sinh^2 A,
\end{aligned}$$

so $\frac{\sin\alpha}{\sinh A} = \frac{1}{\sinh C}$. Interchanging the roles of A and B gives $\frac{\sin\beta}{\sinh B} = \frac{1}{\sinh C}$.

In the general case suppose that the hyperbolic line through the vertex of the triangle with angle α intersects the opposite side at a point which divides that side into segments of lengths A' and A'', so that β is the angle between the sides of lengths C and A'. Suppose also that this hyperbolic line segment has length D. Then the original triangle is divided into two triangles, one with angles $\pi/2, \beta, \alpha'$ and sides of lengths C, D, A' and the other with angles $\pi/2, \gamma, \alpha''$ and sides of lengths B, D, A''. Applying the first part to each of these triangles gives $\sin\beta/\sinh D = 1/\sinh C$ and $\sin\gamma/\sinh D = 1/\sinh B$. Hence, $\sin\beta/\sinh B = \sin\gamma/\sinh C$. The other equation is proved by interchanging the roles of A and B (for example). The case in which the hyperbolic line through a vertex meets

the hyperbolic line through the other two vertices in a point outside the triangle is similar.

11.3.7 (i) This was established in the solution of the preceding exercise.
(ii) Using the sine and cosine rules, $\sin\beta = \frac{\sinh B}{\sinh C}$ and $\cos\alpha = \frac{\cosh B\cosh C - \cosh A}{\sinh B \sinh C}$. Hence, $\frac{\cos\alpha}{\sin\beta} = \frac{\cosh B\cosh C - \cosh A}{\sinh^2 B} = \frac{\cosh^2 B \cosh A - \cosh A}{\sinh^2 B} = \cosh A$.
(iii) Using $\sin\beta = \sinh B/\sinh C$ and the cosine rule for $\cos\beta$, we get $\cot\beta = \frac{\cosh A\cosh C - \cosh B}{\sinh A \sinh B} = \frac{\sinh^2 A \cosh B}{\sinh A\sinh B} = \frac{\sinh A}{\tanh B}$.

11.3.8 If $\gamma = \pi/2$ the formula we want is that in Exercise 11.3.7(ii). In the general case, we use the method (and notation) of the solution of Exercise 11.3.6. In the case where the hyperbolic line through a vertex perpendicular to the opposite side meets that side at a point inside the triangle, applying Exercise 11.3.7(ii) to the two right-angled triangles gives $\cosh A' = \cos\alpha'/\sin\beta$, $\cosh A'' = \cos\alpha''/\sin\gamma$, so

$$\begin{aligned}\cosh A = \cosh(A' + A'') &= \cosh A' \cosh A'' + \sinh A' \sinh A'' \\ &= \frac{\cos\alpha' \cos\alpha''}{\sin\beta\sin\gamma} + \frac{\tanh^2 D}{\tan\beta\tan\gamma}\end{aligned}$$

using Exercise 11.3.7(iii). By Exercise 11.3.7(ii), $\cosh D = \frac{\cos\beta}{\sin\alpha'} = \frac{\cos\gamma}{\sin\alpha''}$ so

$$\begin{aligned}\cosh A \sin\beta\sin\gamma &= \cos\alpha + \sin\alpha'\sin\alpha'' + \tanh^2 D\cos\beta\cos\gamma \\ &= \cos\alpha + \operatorname{sech}^2 D \cos\beta\cos\gamma + \tanh^2 D\cos\beta\cos\gamma \\ &= \cos\alpha + \cos\beta\cos\gamma.\end{aligned}$$

The case in which the perpendicular meets the opposite side at a point outside the triangle is similar.

11.3.9 Let $\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$ be a curve on S^2. Then, $\Pi(\boldsymbol{\gamma}(t)) = (u(t), v(t))$, where $u = \frac{x}{1-z}, v = \frac{y}{1-z}$. Denoting d/dt by a dot, $\dot{u} = \frac{(1-z)\dot{x} + x\dot{z}}{(1-z)^2}$ with a similar formula for $\dot{v}$, which give

$$4\frac{\dot{u}^2 + \dot{v}^2}{1 + u^2 + v^2} = \frac{1}{(1-z)^2}\left((1-z)^2(\dot{x}^2 + \dot{y}^2) + (x^2 + y^2)\dot{z}^2 + 2(x\dot{x} + y\dot{y})\dot{z}(1-z)\right).$$

Using $x^2 + y^2 = 1 - z^2$, which implies $x\dot{x} + y\dot{y} = -z\dot{z}$, this expression simplifies to $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$. Hence, the length of $\Pi\circ\boldsymbol{\gamma}$ calculated using the given first fundamental form on $\mathbb{R}^2$ is $\int \frac{2\sqrt{\dot{u}^2 + \dot{v}^2}}{1 + u^2 + v^2}dt = \int \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}dt$, which is the length of $\boldsymbol{\gamma}$. Hence, Π is an isometry.

11.3.10 Since $\mathcal{P} : \mathcal{H} \to \mathcal{D}_P$ is an isometry, $\mathcal{P}^{-1}(l_1)$ and $\mathcal{P}^{-1}(l_2)$ are geodesics in $\mathcal{H}$. By Proposition 11.2.3, there is an isometry F of $\mathcal{H}$ that takes $\mathcal{P}^{-1}(l_1)$ to $\mathcal{P}^{-1}(l_2)$

and $\mathcal{P}^{-1}(z_1)$ to $\mathcal{P}^{-1}(z_1)$. Then, $\mathcal{P}F\mathcal{P}^{-1}$ is an isometry of $\mathcal{D}_P$ that takes l_1 to l_2 and z_1 to z_2.

11.3.11 Draw the radii of $\mathcal{D}_P$ given by $\arg(z) = 2\pi k/n$ with $k = 0, 1, \ldots, n-1$, and let $a_0, a_1, \ldots, a_{n-1}$ be the points at a Euclidean distance r (with $0 < r < 1$) from the origin on these radii. For each value of r, we then have a regular hyperbolic n-gon with vertices $a_0, a_1, \ldots, a_{n-1}$. It is clear that the interior angle α of this n-gon is an increasing function of r and depends continuously on r. When r tends to zero, the area of the n-gon also tends to zero, so α must approach its Euclidean value (Theorem 11.1.5), namely $(n-2)\pi/n$. On the other hand, as r tends to 1, α tend to zero. By the intermediate value theorem, when $0 < r < 1$, α take every value in the interval $0 < \alpha < (n-2)\pi/n$.

By Exercise 11.3.8,

$$\cosh A = \frac{\cos\frac{2\pi}{n} + \cos^2\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2}}$$

$$\therefore \quad 1 + \cosh A = \frac{2\cos^2\frac{\pi}{n}}{\sin^2\frac{\alpha}{2}}$$

$$\therefore \quad \cosh\frac{A}{2} = \frac{\cos\frac{\pi}{n}}{\sin\frac{\alpha}{2}}.$$

11.3.12 Let D be the length of the diagonal of the quadrilateral, and let β be the angle between the diagonal and the side of length B. By Corollary 11.3.7,

$$\cosh D = \cosh A \cosh B,$$

and by the cosine rule (Theorem 11.3.6),

$$\cosh C = \cosh A \cosh D - \sinh A \sinh D \sin\beta.$$

By the sine rule (Exercise 11.3.6), $\sinh A = \sinh D \sin\beta$ so

$$\cosh C = \cosh A \cosh D - \sinh^2 A = \cosh^2 A \cosh B - \sinh^2 A.$$

By the sine rule, the other two (equal) angles α of the quadrilateral are given by

$$\sin\alpha = \frac{\sinh D \cos\beta}{\sinh C}.$$

By the cosine rule,

$$\cosh A = \cosh B \cosh D - \sinh B \sinh D \cos\beta.$$

Hence,

$$\sin\alpha = \frac{\cosh B \cosh D - \cosh A}{\sinh B \sinh C} = \frac{\cosh A \sinh B}{\sinh C}.$$

11.3.13 Let n be one of the geodesics bisecting the angle between l and m, and let a be a point on n. We will show that a is equidistant from l and m. By applying an isometry we can assume that n is the real axis in $\mathcal{D}_P$.

Let j and k be the geodesics through a meeting l and m at right angles, and let b and c be the points in which j meets l and k meets m, respectively. Reflection R in the real axis is an isometry of $\mathcal{D}_P$ that interchanges l and m and fixes each point of n. If $R(c) \neq b$, the hyperbolic triangle with vertices a, b and $R(c)$ would have two angles equal to $\pi/2$, contradicting Theorem 11.1.5. It follows that R interchanges b and c, and hence that $d(a,b) = d(a,c)$.

Consider a hyperbolic triangle with vertices a, b, c and let l, m, n be the geodesics bisecting the angles at a, b, c, respectively. Let d be the point of intersection of l and m. Then, d is equidistant from all three sides of the triangle, and hence lies on n.

11.3.14 By applying an isometry of $\mathcal{D}_P$, we can assume that l is the imaginary axis, m is the real axis, $a = 0$ and $c = 1$. By Proposition 11.1.4, the hyperbolic circle with centre $b > 0$ and radius R is

$$\frac{|z-b|}{|1-bz|} = \tanh\frac{R}{2}.$$

If this circle passes through the origin, $b = \tanh\frac{R}{2}$ so the hyperbolic circle is

$$|z-b| = b|1-bz|,$$

which can be written as

$$|z|^2 - \frac{b}{1+b^2}(z+\bar{z}) = 0.$$

As $b \to 1$ this tends to the circle $|z|^2 = \frac{1}{2}(z+\bar{z})$, i.e.

$$\left|z - \frac{1}{2}\right|^2 = \frac{1}{4},$$

which is a circle Γ touching $\mathcal{C}$ at 1.

A circle $\tilde{\Gamma}$ that intersects $\mathcal{C}$ perpendicularly at 1 also intersects Γ perpendicularly at 1, and hence the other intersection of Γ and $\tilde{\Gamma}$ is also at right angles.

11.4.1 Let l and m be two distinct hyperbolic lines in $\mathcal{H}$ that do not intersect at any point of $\mathcal{H}$. If l and m are both half-lines they are parallel as they do not have

a common perpendicular. If at least one of l and m is a semicircle, then l and m are parallel if they intersect at a point of the real axis, and ultra-parallel otherwise.

11.4.2 We work in $\mathcal{H}$ and assume that l is the imaginary axis (by applying a suitable isometry). If $a = v + iw$, the semicircle geodesic through a intersects l at $i\sqrt{v^2+w^2} = ir$, say. The distance of a from l is $2\tanh^{-1}\left|\frac{ir-v-iw}{ir-v+iw}\right|$. Setting this equal to a constant, say D, gives (after some algebra) $v^2/w^2 = 2\sinh^2(D/2)$. This is the equation of a pair of lines passing through the origin. As they are not perpendicular to the real axis (unless $D = 0$), they are not hyperbolic lines.

11.4.3 We work in $\mathcal{D}_P$. By applying an isometry we can assume that a is the origin and $b > 0$. Suppose that the hyperbolic triangle with vertices a, b, c has internal angles α, β, γ, and assume that $\mathfrak{Re}(c) > 0$.

The hyperbolic line through b and c is part of a circle Γ with centre d and radius r, say. The line through b and d makes an angle $\pi/2 - \beta$ with the real axis, so $d = b + r\sin\beta + ir\cos\beta$. Similarly, $d = c + r\sin(\alpha+\gamma) - ir\cos(\alpha+\gamma) = c + r\sin(A+\beta) + ir\cos(A+\beta)$ using $A = \pi - \alpha - \beta - \gamma$. Writing $c = v + iw$ we get $v = b - r\sin A\cos\beta + r(1-\cos A)\sin\beta$, $w = r(1-\cos A)\cos\beta + r\sin A\sin\beta$. Then, $v(1-\cos A) + w\sin A = (b + 2r\sin\beta)(1-\cos A)$. But, since γ intersects the boundary $\mathcal{C}$ of $\mathcal{D}_P$ perpendicularly, $r^2 + 1 = |d|^2 = (b + r\sin\beta)^2 + r^2\cos^2\beta = r^2 + b^2 + 2br\sin\beta$, so $2br\sin\beta = 1 - b^2$. Hnece, $v(1-\cos A) + w\sin A = \frac{1-\cos A}{b}$. This is the equation of a straight line that interescts the real axis at $1/b$ and makes an angle $A/2$ with the (negative) real axis. The set of points c for which the triangle with vertices a, b, c has area A is the union of this line together with its reflection in the real axis. These lines are not hyperbolic lines as they do not pass through the origin.

11.4.4 In the first two parts of this exercise it is slightly easier to work in $\mathcal{H}$ (by applying $\mathcal{P}^{-1}$).

(i) By applying a suitable isometry of $\mathcal{H}$, we can assume that one of the sides of the triangle is the imaginary axis. Then the other sides must be a semicircle passing through the origin and some point $a \in \mathbb{R}$, together with the half-line $\mathfrak{Re}(z) = a$. By further applying a suitable translation parallel to the real axis and a positive dilation (which are isometries of $\mathcal{H}$), we can assume that the three sides are the half-lines $\mathfrak{Re}(z) = \pm 1$ and the semicircle $|z|^2 = 1$. Then, the area of the triangle is

$$\int_{-1}^{1} dv \int_{\sqrt{1-v^2}}^{\infty} \frac{dw}{w^2} = \int_{-1}^{1} \frac{dv}{\sqrt{1-v^2}} = \pi.$$

(ii) Let a be the vertex of the biasymptotic triangle in $\mathcal{H}$, and let l be the geodesic through a perpendicular to the opposite side, meeting it at a', say. By applying an isometry of $\mathcal{H}$, we can assume that l is the imaginary axis and that $a' = i$.

Then, the side opposite a is the semicircle $|z|^2 = 1$. The side joining 1 and a is an arc of a circle with centre $b < 0$, say, where

$$\cos\frac{\alpha}{2} = \frac{-b}{1-b}.$$

The equation of the circle is $(v-b)^2 + w^2 = (1-b)^2$, so the area of the triangle is

$$2\int_0^1 \left(\frac{1}{\sqrt{1-v^2}} - \frac{1}{\sqrt{(1-v)(1+v-2b)}}\right) dv.$$

The second integral can be evaluated using the substitution $u = v - b$. This gives the area as

$$\begin{aligned}\pi - 2\int_{-b}^{1-b} \frac{du}{\sqrt{(1-b)^2 - u^2}} du &= 2\sin^{-1}\frac{-b}{1-b} \\ &= \pi - 2\cos^{-1}\left(\frac{-b}{1-b}\right) = \pi - \alpha.\end{aligned}$$

That a bisymptotic triangle with angle α exists for any $0 < \alpha < \pi$ is proved using the method of Exercise 11.3.11.

(iii) We now switch back to $\mathcal{D}_P$. Assume that the vertex at which the angle is α is 0, let b be the other vertex of the triangle in $\mathcal{D}_P$. Let the side of the triangle passing through 0 and b meet $\mathcal{C}$ at c, and let the other side passing through 0 meet $\mathcal{C}$ at d (these sides are radii of $\mathcal{C}$). The area of the asymptotic triangle with vertices $0, b, d$ is the difference between the areas of the biasymptotic triangles with vertices $0, c, d$ and b, c, d, which by part (ii) is

$$(\pi - \alpha) - (\pi - (\pi - \beta)) = \pi - \alpha - \beta.$$

Let a be a point on the radius joining 0 and d and consider the triangle with vertices $0, a, b$. Let the angle of this triangle at b be β' and the angle at a be α'. Thus, α' tends to zero and β' tends to β as a tends to d along the radius ('tends to' in the Euclidean sense). By Exercise 11.3.8, the length A of the side joining 0 and b is given by

$$\cosh A = \frac{\cos\alpha' + \cos\alpha\cos\beta'}{\sin\alpha\sin\beta'}.$$

Letting a tend to c, we obtain

$$\cosh A = \frac{1 + \cos\alpha\cos\beta}{\sin\alpha\sin\beta}.$$

11.4.5 (i) Let $a, b \in \mathcal{D}_P$ be the vertices of one of the triangles, let l be the geodesic passing through a, b and let m be the other side of the triangle passing through a; define a', b', l', m' similarly for the other triangle. By applying an isometry to one of the triangles, we can assume that $a = a'$, $m = m'$ and that l and l' are on the same side of m. As the triangles have the same angle at a, $l = l'$ and by Exercise 11.4.4(iii) the lengths of the finite sides of the two triangles are equal. It follows that $b = b'$, and since the two triangles have the same angle at b their third sides must also coincide.

(ii) As in (i), by applying an isometry to one of the triangles we can assume that their vertices in $\mathcal{D}_P$ coincide, as do the two sides passing through this vertex. But then the two triangles coincide.

(iii) This was established in the solution of Exercise 11.4.4(i).

11.4.6 We consider the biasymptotic triangle with vertices $0, 1$ and $e^{3i\pi/4}$, where the two sides meeting at the origin are radii of $\mathcal{D}_P$ and the side joining 1 and $e^{3i\pi/4}$ is an arc of a circle meeting $\mathcal{C}$ at right angles at 1 and $e^{3i\pi/4}$. The 'altitude' of this triangle passing through 1 is a circular arc that meets $\mathcal{C}$ at $-i$, and that through $e^{3i\pi/4}$ is a circular arc meeting $\mathcal{C}$ at $e^{5i\pi/4}$. These two geodesics are clearly ultraparallel (one is confined to the region $v > 0$, the other to the region $v < 0$). If we now replace the vertices 1 and $e^{3i\pi/4}$ with nearby points $a, b \in \mathcal{D}_P$, the altitudes of the hyperbolic triangle with vertices $1, a, b$ passing through a and b will still be ultraparallel.

11.5.1 From Example 6.3.5, $\Pi^{-1}(v, w) = \left(\frac{2v}{v^2+w^2+1}, \frac{2w}{v^2+w^2+1}, \frac{v^2+w^2-1}{v^2+w^2+1}\right)$, so $\mathcal{K}(v, w) = \frac{2(v,w)}{v^2+w^2+1}$.

11.5.2 From Appendix 2, every Möbius transformation is a composite of transformations of the form $z \mapsto 1/z$, $z \mapsto z + \lambda$, $z \mapsto \lambda z$ (where $\lambda \neq 0$ in the last case). Hence, it suffices to establish Eq. 11.11 when M is of this form. For the last two types this is obvious; for the first, $(a^{-1}, b^{-1}; c^{-1}, d^{-1}) = \frac{(a^{-1}-c^{-1})(b^{-1}-d^{-1})}{(a^{-1}-d^{-1})(b^{-1}-d^{-1})} = (a, b; c, d)$ on multiplying numerator and denominator by $abcd$. This argument is only valid provided none of a, b, c, d is 0 or ∞, but a similar argument works in the other cases. For example, if $a = \infty$ but $b, c, d \neq 0$, we have to show that $(0, b^{-1}; c^{-1}, d^{-1}) = (\infty, b; c, d)$. This is proved by multiplying numerator and denominator of $(0, b^{-1}; c^{-1}, d^{-1}) = \frac{-c^{-1}(b^{-1}-d^{-1})}{-d^{-1}(b^{-1}-c^{-1})}$ by bcd.

If $M : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a bijection satisfying Eq. 11.11, let $b, c, d \in \mathbb{C}_\infty$ be such that $M(b) = 1, M(c) = 0, M(d) = \infty$. Then, $M(z) = (M(z), 1; 0, \infty) = (M(z), M(b); M(c), M(d)) = (z, b; c, d)$, so M is a Möbius transformation.

11.5.3 It is enough to prove the existence when $a' = \infty, b' = 0, c' = 1$. For if M and M' are Möbius transformations taking (a, b, c) and (a', b', c') to $(\infty, 0, 1)$, then ${M'}^{-1} \circ M$ is a Möbius transformation taking (a, b, c) to (a', b', c'). But $M(z) = (a, b; c, z)$ is a Möbius transformation that takes (a, b, c) to $(\infty, 0, 1)$. For

the uniqueness, note that if M is a Möbius transformation that takes (a, b, c) to $(\infty, 0, 1)$, then $M(z) = (\infty, 0; 1, M(z)) = (a, b; c, z)$.

11.5.4 $(a, -1/\bar{a}; b, -1/\bar{b}) = \frac{(a-b)(-\frac{1}{\bar{a}}+\frac{1}{\bar{b}})}{(a+\frac{1}{\bar{b}})(-\frac{1}{\bar{a}}-b)} = \frac{(a-b)(\bar{b}-\bar{a})}{(1+\bar{a}b)(1+a\bar{b})} = -\left|\frac{a-b}{1+\bar{a}b}\right|^2 = -\tan^2\frac{1}{2}d$ by Proposition 6.5.2.

11.5.5 The reflection in the line through the origin making an angle θ with the real axis is $\mathcal{R}(z) = e^{2i\theta}\bar{z}$. Then, $\mathcal{K}(\mathcal{R}(z)) = \frac{2e^{2i\theta}\bar{z}}{|z|^2+1} = e^{2i\theta}\overline{\mathcal{K}(z)} = \mathcal{R}(\mathcal{K}(z))$.

11.5.6 This follows from Exercises 11.3.5 and 11.5.5 and Proposition 11.5.4.

11.5.7 Since $\mathcal{K} = \mathrm{pr} \circ \Pi^{-1}$, and since Π is conformal (Example 6.3.5), it suffices to show that $\mathrm{pr} : \mathcal{S}^2_- \to \mathcal{D}$ preserves the angles between curves intersecting at $\mathrm{pr}(0, 0, -1) = (0, 0, 0)$. But the first fundamental form of the patch $\mathrm{pr}^{-1}(x, y, 0) = (x, y, -\sqrt{1 - x^2 - y^2})$ of $\mathcal{S}^2_-$ is

$$\frac{1}{1 - x^2 - y^2}((1 - y^2)dx^2 + 2xydxdy + (1 - x^2)dy^2),$$

which reduces to $dx^2 + dy^2$ when $x = y = 0$.

11.5.8 Suppose first that a, b, c, d lie on a Circle $\mathcal{C}$. Let M be the Möbius transformation such that $M(a) = \infty$, $M(b) = 0$, $M(c) = 1$ (Exercise 11.5.3), and let $M(d) = z$. Now $M(\mathcal{C})$ is a Circle $\tilde{\mathcal{C}}$, say (Proposition A.2.4); since $\tilde{\mathcal{C}}$ passes through ∞ it must be a straight line, and as this line passes through 0 and 1 it must be the real axis; hence $z \in \mathbb{R}$. But $(a, b; c, d) = (\infty, 0; 1, z) = z$ (Exercise 11.5.2).

Conversely, assume that $(a, b; c, d) \in \mathbb{R}$, and let M be as above. Then, $M(d) \in \mathbb{R}$ and $0, 1, \infty, M(d)$ lie on a Circle $\mathcal{C}$, namely the real axis. But then a, b, c, d lie on the Circle $M^{-1}(\mathcal{C})$.

11.5.9 Let M be the Möbius transformation taking $\infty, 0, 1$ to a, b, c, respectively. Then, $M(d) = \lambda$. By Exercise 11.5.2, if we subject a, b, c, d to any permutation, the resulting cross-ratio will be the cross-ratio of the points obtained by subjecting $\infty, 0, 1, \lambda$ to the *same* permutation. It therefore suffices to check that the six stated values are the values of the cross-ratios of the points $\infty, 0, 1, \lambda$ taken in any order. This is straightforward.

Chapter 12

12.1.1 $\kappa_1 + \kappa_2 = 0 \Longrightarrow \kappa_2 = -\kappa_1 \Longrightarrow K = \kappa_1\kappa_2 = -\kappa_1^2 \leq 0$. $K = 0 \Longleftrightarrow \kappa_1^2 = 0 \Longleftrightarrow \kappa_1 = \kappa_2 = 0 \Longleftrightarrow$ the surface is an open subset of a plane (by Proposition 8.2.9).

12.1.2 From Eq. 8.15, $\boldsymbol{\sigma}^\lambda_u \times \boldsymbol{\sigma}^\lambda_v = (1 - \lambda\kappa_1)(1 - \lambda\kappa_2)\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v$, where κ_1, κ_2 are the principal curvatures of $\boldsymbol{\sigma}$. Since $\boldsymbol{\sigma}$ is minimal, $\kappa_2 = -\kappa_1$ so

$$\begin{aligned}\mathcal{A}_{\boldsymbol{\sigma}^\lambda}(U) &= \int\!\!\int_U (1 - \lambda^2\kappa_1^2) \parallel \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \parallel dudv \\ &= \mathcal{A}_{\boldsymbol{\sigma}}(U) - \lambda^2 \int\!\!\int_U \kappa_1^2 \parallel \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \parallel dudv.\end{aligned}$$

Since the integrand in the last integral is ≥ 0 everywhere, the stated inequality follows. Equality holds $\Longleftrightarrow$ the last integral vanishes, which happens $\Longleftrightarrow$ the integrand vanishes everywhere, i.e. $\Longleftrightarrow \kappa_1 = 0$ everywhere. In that case $\kappa_2 = -\kappa_1 = 0$ also, and $\boldsymbol{\sigma}$ is an open subset of the plane by Proposition 8.2.9.

12.1.3 By Proposition 8.6.1, a compact minimal surface would have $K > 0$ at some point, contradicting Exercise 12.1.1.

12.1.4 The first part follows from Exercises 6.1.2 and 7.1.4. The map which wraps the plane onto the unit cylinder (Example 6.2.4) is a local isometry, but the plane is a minimal surface and the cylinder is not.

12.1.5 If κ_1, κ_2 are the principal curvatures at the point, we have $\kappa_1 + \kappa_2 = 0$ as the surface is minimal, and $\kappa_1 = \kappa_2$ as the point is an umbilic. Hence, $\kappa_1 = \kappa_2 = 0$.

12.1.6 Let Σ^0 be the minimal surface and let $\mathcal{S} = \Sigma^\lambda$ in the notation of Definition 8.5.1. Let k_1, k_2 be the principal curvatures of Σ. By Proposition 8.5.2(ii), we have

$$\frac{1}{\kappa_1} + \frac{1}{\kappa_2} = \frac{1 - \lambda k_1}{\epsilon k_1} + \frac{1 - \lambda k_2}{\epsilon k_2} = -\frac{2\lambda}{\epsilon},$$

since $\frac{1}{k_1} + \frac{1}{k_2} = 0$ (because $k_1 + k_2 = 0$).

12.1.7 The condition for a minimal surface is $LG - 2MF + NE = 0$ (Corollary 8.1.3). From the solution of Exercise 8.1.16, $E = (1 - \kappa a \cos\theta)^2 + \tau^2 a^2$, $F = \tau a^2$, $G = a^2$, $L = a\tau^2 - \kappa \cos\theta(1 - \kappa a \cos\theta)$, $M = \tau a$, $N = a$, and we find that the condition for the tube to be a minimal surface is

$$a(1 - \kappa a \cos\theta)(1 - 2a\cos\theta) = 0,$$

which obviously cannot hold at all points of the surface.

12.1.8 Let $\mathbf{t}_1, \mathbf{t}_2$ be orthonormal principal vectors of $\mathcal{S}$ at $\mathbf{p}$ corresponding to principal curvatures κ_1, κ_2 (Corollary 8.2.2). Then, $\mathbf{t} = a_1\mathbf{t}_1 + a_2\mathbf{t}_2$ for some $a_1, a_2 \in \mathbb{R}$, so $\mathcal{W}(\mathbf{t}) = \kappa_1 a_1 \mathbf{t}_1 + \kappa_2 a_2 \mathbf{t}_2$ (where $\mathcal{W}$ is the Weungarten map of $\mathcal{S}$ at $\mathbf{p}$). Hence,

$$\frac{\langle\langle\langle \mathbf{t}, \mathbf{t} \rangle\rangle\rangle}{\langle \mathbf{t}, \mathbf{t} \rangle} = \frac{\langle \mathcal{W}(\mathbf{t}), \mathcal{W}(\mathbf{t}) \rangle}{\langle \mathbf{t}, \mathbf{t} \rangle} = \frac{\kappa_1^2 a_1^2 + \kappa_2^2 a_2^2}{a_1^2 + a_2^2}.$$

But, as $\mathcal{S}$ is a minimal surface, $\kappa_2 = -\kappa_1$, so

$$\frac{\langle\langle\langle \mathbf{t}, \mathbf{t} \rangle\rangle\rangle}{\langle \mathbf{t}, \mathbf{t} \rangle} = \kappa_1^2 = -\kappa_1\kappa_2 = -K.$$

Alternatively, this exercise can be deduced from Exercise 8.1.6.

12.2.1 By the solution of Exercise 8.1.2, the helicoid $\boldsymbol{\sigma}(u, v) = (v\cos u, v\sin u, \lambda u)$ has

$E = \lambda^2 + v^2, F = 0, G = 1, L = 0, M = \lambda/(\lambda^2 + v^2)^{1/2}, N = 0$, so $H = \frac{LG-2MF+NG}{2(EG-F^2)} = 0$.

12.2.2 A straightforward calculation shows that the first and second fundamental forms of $\boldsymbol{\sigma}^t$ are $\cosh^2 u(du^2 + dv^2)$ and $-\cos t\, du^2 - 2\sin t\, dudv + \cos t\, dv^2$, respectively, so $H = \frac{-\cos t \cosh^2 u + \cos t \cosh^2 u}{2\cosh^4 u} = 0$.

12.2.3 From Example 5.3.1, the cylinder can be parametrized by $\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v\mathbf{a}$, where $\boldsymbol{\gamma}$ is unit-speed, $\| \mathbf{a} \| = 1$ and $\boldsymbol{\gamma}$ is contained in a plane Π perpendicular to $\mathbf{a}$. We have $\boldsymbol{\sigma}_u = \dot{\boldsymbol{\gamma}} = \mathbf{t}$ (a dot denoting d/du), $\boldsymbol{\sigma}_v = \mathbf{a}$, so $E = 1, F = 0, G = 1$; $\mathbf{N} = \mathbf{t} \times \mathbf{a}$, $\boldsymbol{\sigma}_{uu} = \dot{\mathbf{t}} = \kappa\mathbf{n}$, $\boldsymbol{\sigma}_{uv} = \boldsymbol{\sigma}_{vv} = \mathbf{0}$, so $L = \kappa\mathbf{n}.(\mathbf{t} \times \mathbf{a})$, $M = N = 0$. Now $\mathbf{t} \times \mathbf{a}$ is a unit vector parallel to Π and perpendicular to $\mathbf{t}$, hence parallel to $\mathbf{n}$; so $L = \pm\kappa$ and $H = \pm\kappa/2$. So $H = 0 \iff \kappa = 0 \iff \boldsymbol{\gamma}$ is part of a straight line $\iff$ the cylinder is an open subset of a plane.

12.2.4 The first fundamental form is $(\cosh v + 1)(\cosh v - \cos u)(du^2 + dv^2)$, so $\boldsymbol{\sigma}$ is conformal. By Exercise 8.5.1, to show that $\boldsymbol{\sigma}$ is minimal we must show that $\boldsymbol{\sigma}_{uu} + \boldsymbol{\sigma}_{vv} = \mathbf{0}$; but this is so, since $\boldsymbol{\sigma}_{uu} = (\sin u \cosh v, \cos u \cosh v, \sin \frac{u}{2} \sinh \frac{v}{2}) = -\boldsymbol{\sigma}_{vv}$.

(i) $\boldsymbol{\sigma}(0, v) = (0, 1 - \cosh v, 0)$, which is the y-axis. Any straight line is a geodesic.

(ii) $\boldsymbol{\sigma}(\pi, v) = (\pi, 1 + \cosh v, -4\sinh \frac{v}{2})$, which is a curve in the plane $x = \pi$ such that $z^2 = 16\sinh^2 \frac{v}{2} = 8(\cosh v - 1) = 8(y - 2)$, i.e. a parabola. The geodesic equations are $\frac{d}{dt}(E\dot{u}) = \frac{1}{2}E_u(\dot{u}^2 + \dot{v}^2)$, $\frac{d}{dt}(E\dot{v}) = \frac{1}{2}E_v(\dot{u}^2 + \dot{v}^2)$, where a dot denotes the derivative with respect to the parameter t of the geodesic and $E = (\cosh v + 1)(\cosh v - \cos u)$. When $u = \pi$, the unit-speed condition is $E\dot{v}^2 = 1$, so $\dot{v} = 1/(\cosh v + 1)$. Hence, the first geodesic equation is $0 = \frac{1}{2}E_u\dot{v}^2$, which holds because $E_u = \sin u(\cosh v + 1) = 0$ when $u = \pi$; the second geodesic equation is $\frac{d}{dt}(\cosh v + 1) = (\cosh v + 1)\sinh v\, \dot{v}^2 = \sinh v\, \dot{v}$, which obviously holds.

(iii) $\boldsymbol{\sigma}(u, 0) = (u - \sin u, 1 - \cos u, 0)$, which is the cycloid of Exercise 1.1.7 (in the xy-plane, with $a = 1$ and with t replaced by u). The second geodesic equation is satisfied because $E_v = \sinh v(2\cosh v + 1 - \cos u) = 0$ when $v = 0$. The unit-speed condition is $2(1 - \cos u)\dot{u}^2 = 1$, so $\dot{u} = 1/2\sin \frac{u}{2}$. The first geodesic equation is $\frac{d}{dt}(4\sin^2 \frac{u}{2}\dot{u}) = \sin u\dot{u}^2$, i.e. $\frac{d}{dt}(2\sin \frac{u}{2}) = \cos \frac{u}{2}\dot{u}$, which obviously holds.

12.2.5 Using Exercise 8.1.1, the surface is minimal $\iff$

$$(1 + g'^2)\ddot{f} + (1 + \dot{f}^2)g'' = 0,$$

where a dot denotes d/dx and a dash denotes d/dy; hence the stated equation. Since the left-hand side of the stated equation depends only on x and the right-hand side only on y, we must have

$$\frac{\ddot{f}}{1 + \dot{f}^2} = a, \quad \frac{g''}{1 + g'^2} = -a,$$

for some constant a. Suppose that $a \neq 0$. Let $r = \dot{f}$; then $\ddot{f} = rdr/df$ and the first equation is $rdr/df = a(1+r^2)$, which can be integrated to give $af = \frac{1}{2}\ln(1+r^2)$, up to adding an arbitary constant (which corresponds to translating the surface parallel to the z-axis). So $df/dx = \pm\sqrt{e^{2af}-1}$, which integrates to give $f = -\frac{1}{a}\ln\cos a(x+b)$, where b is a constant; we can assume that $b = 0$ by translating the surface parallel to the x-axis. Similarly, $g = \frac{1}{a}\ln\cos ay$, after translating the surface parallel to the y-axis. So, up to a translation, we have

$$az = \ln\left(\frac{\cos ay}{\cos ax}\right),$$

which is obtained from Scherk's surface by the dilation $(x, y, z) \mapsto a(x, y, z)$. If $a = 0$, then $\ddot{f} = g'' = 0$ so $f = b + cx$, $g = d + ey$, for some constants b, c, d, e, and we have the plane $z = b + d + cx + ey$.

12.2.6 Using Exercise 8.1.1, taking $f(x, y) = \sin^{-1}(\sinh x \sinh y)$, and writing $\lambda = 1/\sqrt{1 - \sinh^2 x \sinh^2 y}$, we find that $f_x = \lambda \cosh x \sinh y$, $f_y = \lambda \cosh y \sinh x$, $f_{xx} = \lambda^3 \sinh x \sinh y \cosh^2 y$, $f_{xy} = \lambda^3 \cosh x \cosh y$, $f_{yy} = \lambda^3 \sinh x \sinh y \cosh^2 x$. It is easy to check that $(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0$.

12.3.1 (i) From the proof of Theorem 12.3.2, the Gauss map is conformal $\iff \mathcal{W}^2 = \lambda.\mathrm{id}$, where λ is a smooth function on $\mathcal{S}$. If $H \neq 0$ at $\mathbf{p}$, then $H \neq 0$ on an open subset $\mathcal{O}$ of $\mathcal{S}$ containing $\mathbf{p}$. By Exercise 8.1.6, $\mathcal{W} = \frac{\lambda^2+K}{2H}.\mathrm{id}$, so every point of $\mathcal{O}$ is an umbilic and $\mathcal{O}$ is an open subset of a plane or a sphere by Proposition 8.2.9. Since $H \neq 0$ the planar case is impossible.

Part (ii) is now obvious.

For (iii), assume that $\mathcal{S}$ is not minimal. Then, there is a point $\mathbf{p} \in \mathcal{S}$ at which $H \neq 0$, say $H = \mu$. The argument in (i) and (ii) shows that the set $\mathcal{S}_\mu$ of points of $\mathcal{S}$ at which $H = \mu$ is a (non-empty) open subset of $\mathcal{S}$; it is also a closed subset because H is a continuous function on $\mathcal{S}$. Since $\mathcal{S}$ is connected, $\mathcal{S}_\mu = \mathcal{S}$. Hence, every point of $\mathcal{S}$ is an umbilic, and so $\mathcal{S}$ is an open subset of a sphere (the planar case is impossible as $\mu \neq 0$).

12.3.2 (i) From Example 12.1.4, $\mathbf{N} = (-\operatorname{sech} u \cos v, -\operatorname{sech} u \sin v, \tanh u)$. Hence, if $\mathbf{N}(u, v) = \mathbf{N}(u', v')$, then $u = u'$ since $u \mapsto \tanh u$ is injective, so $\cos v = \cos v'$ and $\sin v = \sin v'$, hence $v = v'$; thus, $\mathbf{N}$ is injective. If $\mathbf{N} = (x, y, z)$, then $x^2+y^2 = \operatorname{sech}^2 u \neq 0$, so the image of $\mathbf{N}$ does not contain the poles. Given a point $(x, y, z) \in S^2$ other than the poles, let $u = \pm\operatorname{sech}^{-1}\sqrt{x^2+y^2}$, the sign being that of z, and let v be such that $\cos v = -x/\sqrt{x^2+y^2}$, $\sin v = -y/\sqrt{x^2+y^2}$; then, $\mathbf{N}(u, v) = (x, y, z)$.
(ii) By the solution of Exercise 8.1.2, $\mathbf{N} = (\lambda^2 + v^2)^{-1/2}(-\lambda \sin u, \lambda \cos u, -v)$. Since $\mathbf{N}(u, v) = \mathbf{N}(u + 2k\pi, v)$ for all integers k, the infinitely-many points

$$\boldsymbol{\sigma}(u + 2k\pi, v) = \boldsymbol{\sigma}(u, v) + (0, 0, 2k\pi)$$

of the helicoid all have the same image under the Gauss map. (Of course, this is geometrically obvious because the helicoid itself is left unchanged by the translation by 2π parallel to the z-axis.) If $\mathbf{N} = (x, y, z)$, then $x^2+y^2 = \lambda^2/(\lambda^2+v^2) \neq 0$, so the image of $\mathbf{N}$ does not contain the poles. If $(x, y, z) \in S^2$ and $x^2 + y^2 \neq 0$, let $v = -\lambda z/\sqrt{x^2+y^2}$ and let u be such that $\sin u = -x/\sqrt{x^2+y^2}$, $\cos u = -y/\sqrt{x^2+y^2}$; then $\mathbf{N}(u, v) = (x, y, z)$.

12.4.1 By Proposition 12.3.2, if $K \neq 0$ then the Gauss map $\mathcal{G} : \mathcal{S} \to S^2$ is a conformal local diffeomorphism. Let R be a rotation of $\mathbb{R}^3$ about the origin that takes $\mathcal{G}(\mathbf{p})$ to the south pole of S^2 (or any point other than the north pole). There is an open subset $\mathcal{O}$ of $\mathcal{S}$ containing $\mathbf{p}$ such that $\mathcal{G}(\mathcal{O})$ does not contain the north pole. By Example 6.3.5, $\Pi \circ R \circ \mathcal{G}$ is a conformal diffeomorphism from $\mathcal{O}$ to an open subset U of $\mathbb{R}^2$. The inverse of this diffeomorphism is the desired surface patch $\boldsymbol{\sigma}$.

12.4.2 Writing $x = r\cos\theta$, $y = r\sin\theta$ gives $z = a\theta$, so we have the parametrization $\boldsymbol{\sigma}(r, \theta) = (r\cos\theta, r\sin\theta, a\theta)$. The first fundamental form of $\boldsymbol{\sigma}$ is $dr^2+(r^2+a^2)d\theta^2$, so $\boldsymbol{\sigma}$ is not a conformal parametrization. However, noting that

$$\int \frac{dr}{\sqrt{r^2+a^2}} = \sinh^{-1}\frac{r}{a},$$

we let $\varphi = \sinh^{-1}\frac{r}{a}$ so that $r = a\sinh\varphi$. The reparametrization

$$\tilde{\boldsymbol{\sigma}}(\theta, \varphi) = \boldsymbol{\sigma}(a\sinh\varphi, \theta) = (a\sinh\varphi\cos\theta, a\sinh\varphi\sin\theta, a\theta)$$

has first fundamental form

$$(r^2+a^2)(d\varphi^2+d\theta^2) = a^2\cosh^2\varphi(d\theta^2+d\varphi^2),$$

so $\tilde{\boldsymbol{\sigma}}$ is a conformal parametrization of the surface.

12.5.1 We have

$$\begin{aligned}
\boldsymbol{\varphi} &= \boldsymbol{\sigma}_u - i\boldsymbol{\sigma}_v \\
&= (1-u^2+v^2-2iuv, 2uv - i(1-v^2+u^2), 2u+2iv) \\
&= (1-\zeta^2, -i(1+\zeta^2), 2\zeta).
\end{aligned}$$

So the conjugate surface is, up to a translation,

$$\begin{aligned}
\tilde{\boldsymbol{\sigma}}(u, v) &= \mathfrak{Re}\int (i(1-\zeta^2), 1+\zeta^2, 2i\zeta)\, d\zeta \\
&= \mathfrak{Re}\left(i\left(\zeta - \frac{\zeta^3}{3}\right), \zeta + \frac{\zeta^3}{3}, i\zeta^2\right) \\
&= \left(-v + u^2 v - \frac{v^3}{3}, u + \frac{u^3}{3} - uv^2, -2uv\right).
\end{aligned}$$

Let $U = (u-v)/\sqrt{2}$, $V = (u+v)/\sqrt{2}$, $\tilde{\boldsymbol{\sigma}}(U,V) = \boldsymbol{\sigma}(u,v)$; then,

$$\tilde{\boldsymbol{\sigma}}(U,V) = \Bigg(\frac{1}{\sqrt{2}}\left(U - V + UV^2 - U^2V + \frac{1}{3}V^3 - \frac{1}{3}U^3\right), \\ \frac{1}{\sqrt{2}}\left(U + V + UV^2 + U^2V - \frac{1}{3}V^3 - \frac{1}{3}U^3\right), U^2 - V^2\Bigg).$$

Applying the $\pi/4$ rotation $(x,y,z) \mapsto \left(\frac{1}{\sqrt{2}}(x+y), \frac{1}{\sqrt{2}}(y-x), z\right)$ to $\tilde{\boldsymbol{\sigma}}(U,V)$ then gives $\left(U - \frac{1}{3}U^3 + UV^2, V - \frac{1}{3}V^3 + U^2V, U^2 - V^2\right)$, which is Enneper's surface again.

12.5.2 $\boldsymbol{\varphi} = \left(\frac{1}{2}(1-\zeta^{-4})(1-\zeta^2), \frac{i}{2}(1-\zeta^{-4})(1+\zeta^2), \zeta(1-\zeta^{-4})\right)$, so

$$\begin{aligned}\boldsymbol{\sigma} &= \mathfrak{Re}\left(\frac{1}{2}\left(\zeta - \frac{\zeta^3}{3} - \zeta^{-1} + \frac{\zeta^{-3}}{3}\right), \frac{i}{2}\left(\zeta + \frac{\zeta^3}{3} + \zeta^{-1} + \frac{\zeta^{-3}}{3}\right), \frac{\zeta^2}{2} + \frac{\zeta^{-2}}{2}\right) \\ &= \mathfrak{Re}\left(-\frac{1}{6}(\zeta - \zeta^{-1})^3, \frac{i}{6}(\zeta + \zeta^{-1})^3, \frac{1}{2}(\zeta + \zeta^{-1})^2\right),\end{aligned}$$

up to a translation. Put $\zeta = e^{\tilde{\zeta}}$, $\tilde{\zeta} = \tilde{u} + i\tilde{v}$. Then, $\boldsymbol{\sigma}(u,v) = \tilde{\boldsymbol{\sigma}}(\tilde{u},\tilde{v})$, where

$$\begin{aligned}\tilde{\boldsymbol{\sigma}}(\tilde{u},\tilde{v}) &= \mathfrak{Re}\left(-\frac{4}{3}\sinh^3\tilde{\zeta}, \frac{4i}{3}\cosh^3\tilde{\zeta}, 2\cosh^2\tilde{\zeta}\right) \\ &= \Bigg(4\sinh\tilde{u}\cos\tilde{v}(\cosh^2\tilde{u}\sin^2\tilde{v} - \frac{1}{3}\sinh^2\tilde{u}\cos^2\tilde{v}), \\ &\qquad 4\sinh\tilde{u}\sin\tilde{v}(\frac{1}{3}\sinh^2\tilde{u}\sin^2\tilde{v} - \cosh^2\tilde{u}\cos^2\tilde{v}), \\ &\qquad 2(\cosh^2\tilde{u}\cos^2\tilde{v} - \sinh^2\tilde{u}\sin^2\tilde{v})\Bigg).\end{aligned}$$

12.5.3 The first part is obvious.
(i) If $a \in \mathbb{R}$, the identity $\boldsymbol{\sigma}^a_u - i\boldsymbol{\sigma}^q_v = a(\boldsymbol{\sigma}_u - i\boldsymbol{\sigma}_v)$ implies that $\boldsymbol{\sigma}^a = a\boldsymbol{\sigma} + \mathbf{a}$, where $\mathbf{a}$ is a constant vector. Hence, $\boldsymbol{\sigma}^a$ is obtained from $\boldsymbol{\sigma}$ by applying the dilation D_a followed by the translation $T_{\mathbf{a}}$ (Appendix 1).
(ii) If f and g are the functions in the Weierstrass representation of $\boldsymbol{\sigma}$ (Proposition 12.5.4), those in the Weierstrass representation of $\boldsymbol{\sigma}^a$ are af and g (see Eq. 12.23). By Eq. 12.25, replacing f by af leaves the first fundamental form unchanged, so the map $\boldsymbol{\sigma}(u,v) \mapsto \boldsymbol{\sigma}^a(u,v)$ is an isometry, and by Eq. 12.26 $\mathbf{N}$ does not depend on f, so the tangent planes of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^a$ at corresponding points are parallel.

12.5.4 We have $\boldsymbol{\sigma}^{e^{it}}_u - i\boldsymbol{\sigma}^{e^{it}}_v = e^{it}(\boldsymbol{\sigma}_u - i\boldsymbol{\sigma}_v)$. Since $\boldsymbol{\sigma}(u,v) = \cosh u\cos v, \cosh u\sin v, u)$, we get

$$\begin{aligned}\boldsymbol{\sigma}^{e^{it}}_u &= (\cos t\sinh u\cos v - \sin t\cosh u\sin v, \sin t\cosh u\cos v + \cos t\sinh u\sin v, \cos t), \\ \boldsymbol{\sigma}^{e^{it}}_v &= (-\cos t\cosh u\sin v - \sin t\sinh u\cos v, \cos t\cosh u\cos v - \sin t\sinh u\sin v, -\sin t).\end{aligned}$$

Integrating gives

$$\begin{aligned}\boldsymbol{\sigma}^{e^{it}}(u,v) &= \cos t(\cosh u\cos v, \cosh u\sin v, u) + \sin t(-\sinh u\sin v, \sinh u\cos v, -v)\\ &= \cos t\boldsymbol{\sigma}(u,v) + \sin t\hat{\boldsymbol{\sigma}}(u,v),\end{aligned}$$

say (up to a translation). In the notation of Exercise 6.2.3, $\tilde{\boldsymbol{\sigma}}(\sinh u, \frac{\pi}{2}+v) = (-\sinh u\sin v, \sinh u\cos v, \frac{\pi}{2}+v)$. Reflecting in the xy-plane and then translating by $\pi/2$ along the z-axis takes $\tilde{\boldsymbol{\sigma}}(\sinh u, \frac{\pi}{2}+v)$ to $\hat{\boldsymbol{\sigma}}(u,v)$.

12.5.5 (i) $\boldsymbol{\varphi}$ is never zero since we have arranged that F' and G' are never both zero. Condition (ii) in Theorem 12.5.2 is obvious.
(ii) When $v=0$, $F'(z) = \frac{\partial}{\partial u}F(u,0) = F_u$, etc, so $\frac{d}{du}\boldsymbol{\sigma}(u,0) = (\dot f, \dot g, 0) = \dot{\boldsymbol{\gamma}}$ (a dot denoting d/du). This proves (ii).
(iii) When $v=0$, $\boldsymbol{\varphi} = (\dot f, \dot g, i\sqrt{\dot f^2+\dot g^2})$. Using Eq. 12.26, $\mathbf{N} = \frac{(-\dot g, \dot f, 0)}{\sqrt{\dot f^2+\dot g^2}}$. Then, $\mathbf{N}\times\dot{\boldsymbol{\gamma}} = (0,0,-\sqrt{\dot f^2+\dot g^2})$ and finally $\ddot{\boldsymbol{\gamma}}.(\mathbf{N}\times\dot{\boldsymbol{\gamma}}) = 0$. It follows that $\boldsymbol{\gamma}$ is a pre-geodesic on $\boldsymbol{\sigma}$ (Exercise 9.1.2).
(iv) If $\boldsymbol{\gamma}$ is the cycloid, $F(z) = z - \sin z, G(z) = 1-\cos z$, so $\boldsymbol{\sigma}_u - i\boldsymbol{\sigma}_v = \boldsymbol{\varphi} = (1-\cos z, \sin z, 2i\sin z)$. This gives

$$\begin{aligned}\boldsymbol{\sigma}_u &= (1-\cos u\cosh v, \sin u\cosh v, -2\cos\frac{u}{2}\sinh\frac{v}{2}),\\ \boldsymbol{\sigma}_v &= (-\sin u\sinh v, -\cos u\sinh v, -2\sin\frac{u}{2}\cosh\frac{v}{2}).\end{aligned}$$

Integrating gives $\boldsymbol{\sigma}(u,v) = (u - \sin u\cosh v, -\cos u\cosh v, -4\sin\frac{u}{2}\sinh\frac{v}{2})$, up to a translation. Translating by $(0,1,0)$ gives Catalan's surface.

12.5.6 $\boldsymbol{\varphi} = (\frac{1}{2}f(1-g^2), \frac{i}{2}f(1+g^2), fg) \Longrightarrow i\boldsymbol{\varphi} = (\frac{1}{2}if(1-g^2), \frac{i}{2}if(1+g^2), ifg)$, which corresponds to the pair if and g.

12.5.7 From Example 12.5.3, for the catenoid we have $\varphi_1 = \sinh\zeta$, $\varphi_2 = -i\cosh\zeta$, $\varphi_3 = 1$, so $f = \varphi_1 - i\varphi_2 = \sinh\zeta - \cosh\zeta = -e^{-\zeta}$, $g = \varphi_3/(\varphi_1 - i\varphi_2) = -e^{\zeta}$. Since the helicoid is the conjugate surface of the catenoid (Example 12.5.3 again), the preceding exercise gives $f = -ie^{-\zeta}$, $g = -e^{\zeta}$ for the helicoid.

12.5.8 By Example 6.3.5, $\Pi(x,y,z) = (x+iy)/(1-z)$. From Eq. 12.26, $\mathcal{G}(\zeta) = \frac{1}{|g|^2+1}(g+\bar g, -i(g-\bar g), |g|^2-1)$, so

$$\pi(\mathcal{G}(\zeta)) = \frac{g+\bar g + g - \bar g}{|g|^2+1-(|g|^2-1)} = g(\zeta).$$

12.5.9 By Eq. 12.22,

$$\boldsymbol{\varphi} = \left(-\frac{1}{2}(\zeta^2-\zeta^{-2}), \frac{i}{2}(\zeta^2+\zeta^{-2}), 1\right),$$

so the parametrization is, up to a translation,

$$\begin{aligned}\boldsymbol{\sigma}(u,v) &= \mathfrak{Re}\int \boldsymbol{\varphi}(\zeta)d\zeta \\ &= \mathfrak{Re}\left(-\frac{(u+iv)^3}{6} - \frac{u-iv}{2(u^2+v^2)}, \frac{i(u+iv)^3}{6} - \frac{i(u-iv)}{2(u^2+v^2)}, u+iv\right) \\ &= \left(-\frac{1}{6}(u^3-3uv^2) - \frac{u}{2(u^2+v^2)}, \frac{1}{6}(-3u^2v+v^3) - \frac{v}{2(u^2+v^2)}, u\right).\end{aligned}$$

By Proposition 12.5.5, the Gaussian curvature is

$$K = \frac{-64|\zeta|^6}{(1+|\zeta|^4)^4},$$

and this tends to zero as $|\zeta| = \sqrt{u^2+v^2} \to \infty$.

12.5.10 Using the conformal parametrization in Exercise 12.4.2, we get

$$\begin{aligned}\boldsymbol{\varphi} &= \boldsymbol{\sigma}_\theta - i\boldsymbol{\sigma}_\varphi \\ &= a(-\sinh\varphi\sin\theta - i\cosh\varphi\cos\theta, \sinh\varphi\cos\theta - i\cosh\varphi\sin\theta, 1) \\ &= a(-i\cos\zeta, -i\sin\zeta, 1),\end{aligned}$$

where $\zeta = \theta + i\varphi$. Then,

$$\begin{aligned}f &= \varphi_1 - i\varphi_2 = -iae^{-i\zeta}, \\ g &= \frac{\varphi_3}{\varphi_1 - i\varphi_2} = ie^{i\zeta}.\end{aligned}$$

By Proposition 12.5.5,

$$\begin{aligned}K &= -\frac{16|e^{i\zeta}|^2}{a^2|e^{-i\zeta}|^2(1+|e^{i\zeta}|^2)^4} \\ &= -\frac{16e^{-2\varphi}}{a^2e^{2\varphi}(1+e^{-2\varphi})^4} \\ &= -\frac{16}{a^2(e^\varphi+e^{-\varphi})^4} \\ &= -a^{-2}\mathrm{sech}^4\varphi.\end{aligned}$$

Chapter 13

13.1.1 If $\boldsymbol{\gamma}$ is a *simple* closed geodesic, Theorem 13.1.2 gives $\iint_{\text{int}(\boldsymbol{\gamma})} K d\mathcal{A} = 2\pi$; since $K \leq 0$, this is impossible. The parallels of a cylinder are not the images under a surface patch $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ of a *simple* closed curve $\boldsymbol{\pi}$ in the plane such that $\text{int}(\boldsymbol{\pi})$ is contained in U. Note that the whole cylinder can actually be covered by a single patch (see Exercise 4.1.4) in which U is an annulus, but that the parallels correspond to circles going 'around the hole' in the annulus.

13.1.2 Since the unit normal $\mathbf{N}$ of S^2 is equal to $\pm\mathbf{n}$, the geodesic curvature κ_g of $\mathbf{n}$ is, up to a sign, $\mathbf{n}''.(\mathbf{n} \times \mathbf{n}')$. Let t be the arc-length of $\boldsymbol{\gamma}$ and denote d/dt by a dot. Then, $ds/dt = \parallel \dot{\mathbf{n}} \parallel = \parallel -\kappa\mathbf{t} + \tau\mathbf{b} \parallel = \sqrt{\kappa^2 + \tau^2} = R$, say, where $\mathbf{t} = \dot{\boldsymbol{\gamma}}$. Then, $\mathbf{n}' = (-\kappa\mathbf{t} + \tau\mathbf{b})/R$, $\mathbf{n} \times \mathbf{n}' = (\kappa\mathbf{b} + \tau\mathbf{t})/R$, and $\mathbf{n}'' = \frac{1}{R}\frac{d}{dt}\left(\frac{-\kappa\mathbf{t}+\tau\mathbf{b}}{R}\right) = -R^{-1}(\kappa/R)^{\cdot}\mathbf{t} + R^{-1}(\tau/R)^{\cdot}\mathbf{b} - R^{-2}(\kappa^2 + \tau^2)\mathbf{n}$. These formulas give

$$\mathbf{n}''.(\mathbf{n} \times \mathbf{n}') = -R^{-2}\tau(\kappa/R)^{\cdot} + R^{-2}\kappa(\tau/R)^{\cdot} = (\kappa\dot{\tau} - \tau\dot{\kappa})/R^3.$$

Since $\dot{\kappa} = R\kappa'$, etc, $\kappa_g = \pm\frac{\kappa\tau' - \tau\kappa'}{\kappa^2+\tau^2} = \pm\frac{d}{ds}\tan^{-1}\frac{\tau}{\kappa}$. Applying Theorem 13.1.2 to the curve $\mathbf{n}$ on S^2, and noting that $K = 1$ for S^2 and that $\int_0^{\ell(\mathbf{n})} \kappa_g \, dt = 0$ because κ_g is the derivative of an $\ell(\mathbf{n})$-periodic function (where $\ell(\mathbf{n})$ is the length of the closed curve $\mathbf{n}$), we get that the area inside $\mathbf{n}$ is 2π.

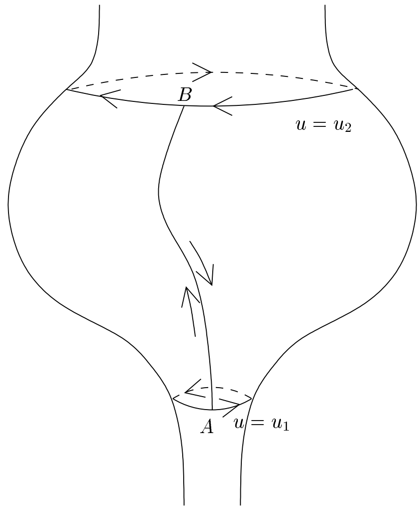
13.1.3 Before smoothing $K = 0$ so the total curvature is zero. To compute the total curvature after smoothing, we apply Theorem 13.1.2 to the smoothed cone with $\boldsymbol{\gamma}$ being the circle on the cone given by $z = a$ for some constant $a > 0$. The radius of this circle is $a \tan\alpha$ so its curvature is $\kappa = 1/a\tan\alpha$. Since the angle of the cone is α, the geodesic curvature of $\boldsymbol{\gamma}$ is $\kappa_g = \kappa\sin\alpha = a^{-1}\cos\alpha$. The part of the cone with $z > a$ still has $K = 0$ so does not contribute to the total curvature. Integrating K over the part of the cone with $z < a$, Gauss-Bonnet gives the total curvature as

$$\iint K d\mathcal{A} = 2\pi - \int_{\boldsymbol{\gamma}} \kappa_g \, ds = 2\pi - (a^{-1}\cos\alpha)(2\pi a \tan\alpha) = 2\pi(1 - \sin\alpha).$$

13.2.1 The parallel $u = u_1$ is the circle $\boldsymbol{\gamma}_1(v) = (f(u_1)\cos v, f(u_1)\sin v, g(u_1))$; if s is the arc-length of $\boldsymbol{\gamma}_1$, $ds/dv = f(u_1)$. Denote d/ds by a dot and d/du by a dash. Then, $\dot{\boldsymbol{\gamma}} = (-\sin v, \cos v, 0)$, $\ddot{\boldsymbol{\gamma}} = -\frac{1}{f(u_1)}(\cos v, \sin v, 0)$, and the unit normal of the surface is $\mathbf{N} = (-g'\cos v, -g'\sin v, f')$. This gives the geodesic curvature of $\boldsymbol{\gamma}$ as $\kappa_g = \ddot{\boldsymbol{\gamma}}.(\mathbf{N} \times \dot{\boldsymbol{\gamma}}) = \frac{f'(u_1)}{f(u_1)}$. Since $\ell(\boldsymbol{\gamma}_1) = 2\pi f(u_1)$, $\int_0^{\ell(\boldsymbol{\gamma}_1)} \kappa_g \, ds = 2\pi f'(u_1)$. Similarly for $\boldsymbol{\gamma}_2$. By Example 8.1.4, $K = -f''/f$, so $\iint_R K d\mathcal{A}_{\boldsymbol{\sigma}} = \int_0^{2\pi}\int_{u_1}^{u_2} -\frac{f''}{f} f \, du dv = 2\pi(f'(u_1) - f'(u_2))$. Hence,

$$\int_0^{\ell(\boldsymbol{\gamma}_1)} \kappa_g \, ds - \int_0^{\ell(\boldsymbol{\gamma}_2)} \kappa_g \, ds = \iint_R K d\mathcal{A}_{\boldsymbol{\sigma}}.$$

This equation is the result of applying Theorem 13.2.2 to the curvilinear polygon shown below.



13.2.2 Applying Theorem 12.2.2 to the curvilinear n-gon, and noting the $-K \geq 1$, we get

$$\mathcal{A}(\mathrm{int}(\boldsymbol{\gamma})) \leq (n-2)\pi - \sum_i \alpha_i.$$

If $n \leq 2$ the right-hand side is < 0, which is absurd. If $n = 3$ the right hand-side is $< \pi$.

13.2.3 Consider a small curvilinear quadrilateral on $\mathcal{S}$ bounded by parameter curves $u = a$, $u = a + h$, $v = b$, $v = b + k$, say, where h and k are small. The internal angles are $\alpha, \alpha, \pi - \alpha, \pi - \alpha$, where α is the angle between the parameter curves $u = a$ and $v = b$. Let $\mathcal{A}$ be the (small) area of the quadrilateral. Since the sides of the quadrilateral are geodesics, Theorem 13.2.2 gives

$$K(a, b)\mathcal{A} = (4-2)\pi - (\alpha + \alpha + \pi - \alpha + \pi - \alpha) = 0.$$

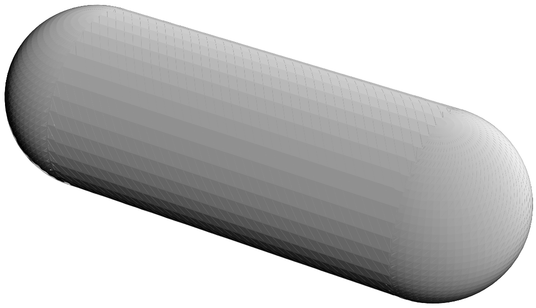
13.3.1 This can be proved by expressing $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in component form and computing both sides. Alternatively, one may observe that both sides of the equation are linear in each of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ (separately), and change sign when any two of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are interchanged. This means that it is enough to prove the formula when $\mathbf{a} = \mathbf{i}, \mathbf{b} = \mathbf{j}, \mathbf{c} = \mathbf{k}$, when both sides are obviously equal to 1.

13.3.2 Define φ_k as in the hint. The stated properties are easily checked.

13.4.1 By Corollary 13.4.8, $\iint_{\mathcal{S}} K d\mathcal{A} = 4\pi(1-g)$, and $g = 1$ since $\mathcal{S}$ is diffeomorphic to T_1. By Proposition 8.6.1, $K > 0$ at some point of $\mathcal{S}$.

13.4.2 $K > 0 \Longrightarrow \iint_{\mathcal{S}} K d\mathcal{A} > 0 \Longrightarrow g < 1$ by Corollary 13.4.8; since g is a non-negative integer, $g = 0$ so $\mathcal{S}$ is diffeomorphic to a sphere. The converse is false: for

example, a 'cigar tube' is diffeomorphic to a sphere but $K = 0$ on the cylindrical part.



13.4.3 The ellipsoid is diffeomorphic to S^2 by the map $(x, y, x) \mapsto (x/a, y/a, z/b)$, so the genus of the ellipsoid is zero. Hence, Corollary 13.4.8 gives $\iint_{\mathcal{S}} K d\mathcal{A} = 4\pi(1-0) = 4\pi$.

Parametrising the ellipsoid by $\boldsymbol{\sigma}(\theta, \varphi) = (a\cos\theta\cos\varphi, a\cos\theta\sin\varphi, b\sin\theta)$ (cf. the latitude-longitude parametrization of S^2), the first and second fundamental forms of $\boldsymbol{\sigma}$ are

$$(a^2\sin^2\theta + b^2\cos^2\theta)d\theta^2 + a^2\cos^2\theta d\varphi^2 \text{ and } \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}(d\theta^2 + \cos^2\theta d\varphi^2),$$

respectively. This gives

$$K = \frac{b^2}{(a^2\sin^2\theta + b^2\cos^2\theta)^2},$$
$$d\mathcal{A}_{\boldsymbol{\sigma}} = a\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta d\varphi.$$

Hence,

$$\iint_{\mathcal{S}} K d\mathcal{A}_{\boldsymbol{\sigma}} = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \frac{ab^2\cos\theta\, d\theta d\varphi}{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}.$$

13.4.4 The surface in Exercise 5.4.1(ii) is diffeomorphic to S^2, hence its Euler number is 2.

13.5.1 Assuming that every country has ≥ 6 neighbours, the argument in the proof of Theorem 13.5.1 gives $E \geq 3F$ and $2E \geq 3V$, so $V - E + F \leq \frac{2E}{3} - E + \frac{E}{3} = 0$, contradicting $V - E + F = 2$.

13.5.2 $3F = 2E$ because each face has three edges and each edge is an edge of two faces. From $\chi = V - E + F$, we get $\chi = V - E + \frac{2}{3}E$, so $E = 3(V - \chi)$.

Since each edge has two vertices and two edges cannot intersect in more than one vertex, $E \leq \frac{1}{2}V(V-1)$; hence, $3(V-\chi) \leq \frac{1}{2}V(V-1)$, which is equivalent to $V^2 - 7V + 6\chi \geq 0$. The roots of the quadratic are $\frac{1}{2}\left(7 \pm \sqrt{49-7\chi}\right)$, so $V \leq \frac{1}{2}\left(7 - \sqrt{49-7\chi}\right)$ or $V \geq \frac{1}{2}\left(7 + \sqrt{49-7\chi}\right)$. Since $\chi = 2, 0, -2, \ldots$, the first condition gives $V \leq 3$, which would allow only one triangle; hence, the second condition must hold.

13.5.3 This is obvious since a diffeomorphism $\mathcal{S} \to \tilde{\mathcal{S}}$ takes an n-coloured map on $\mathcal{S}$ to an n-coloured map on $\tilde{\mathcal{S}}$.

13.5.4 Since three edges meet at each vertex and each edge has two vertices, we have $3V = 2E$. As each edge is an edge of two countries,

$$2E = \sum_{n=2}^{\infty} nc_n.$$

Hence, $E = \frac{1}{2}\sum nc_n$, $V = \frac{1}{3}\sum nc_n$, and since $F = \sum c_n$ we get

$$\chi = \sum_{n=2}^{\infty}\left(\frac{1}{3}n - \frac{1}{2}n + 1\right)c_n = \frac{1}{6}\sum_{n=2}^{\infty}(6-n)c_n.$$

13.5.5 (i) In the notation of the preceding exercise, for a soccer ball we have $c_n = 0$ unless $n = 5$ or 6 so the formula in the preceding exercise gives $c_5 = 12$ (since $\chi = 2$ for a sphere).
(ii) This time we are given that $c_n = 0$ unless $n = 4$ or 6, so the formula gives $2c_4 = 12$.

13.6.1 The circle $\theta = \theta_0$ is a circle in the plane $z = b\sin\theta_0$ with centre on the z-axis, so its principal normal is a unit vector perpendicular to the circle and in this plane, hence equal (up to a sign) to $(\cos\varphi, \sin\varphi, 0)$. The unit normal of $\boldsymbol{\sigma}$ is $\mathbf{N} = (-\cos\theta\cos\varphi, -\cos\theta\sin\varphi, -\sin\theta)$, so the angle between $\mathbf{N}$ and $\mathbf{n}$ at a point of the circle $\theta = \theta_0$ is θ_0. The radius of the circle is $a + b\cos\theta_0$, so its geodesic curvature is $\frac{\sin\theta_0}{a+b\cos\theta_0}$. Hence, $\int \kappa_g ds = 2\pi\sin\theta_0$ and the holonomy is $2\pi - 2\pi\sin\theta_0$. The circles $\varphi =$ constant are geodesics (as they are meridians on a surface of revolution) so $\kappa_g = 0$ and the holonomy is $2\pi - 0 = 2\pi$.

13.6.2 For the circle $v = 1$ on the cone, the radius is 1 and the angle between the principal normal $\mathbf{n}$ of the circle and the unit normal $\mathbf{N}$ of the cone is $\pi/4$, so the holonomy is $2\pi - 2\pi/\sqrt{2} = (2-\sqrt{2})\pi$. The cone is flat so if the converse of Proposition 13.6.5 were true the holonomy around any closed curve on the cone would be zero.

13.6.3 For a general closed curve, $\int_0^{\ell(\boldsymbol{\gamma})} \frac{d\varphi}{ds} ds$ is an integer multiple of 2π, say $2n\pi$. Then, (13.28) is replaced by

$$2n\pi - \int_0^{\ell(\boldsymbol{\gamma})} \kappa_g \, ds.$$

13.6.4 As in the proof of Proposition 13.6.1, $\mathbf{v} = \cos\varphi\mathbf{t} + \sin\theta\mathbf{t}_1$, where $\mathbf{t}_1 = \mathbf{N} \times \mathbf{t}$, so

$$\dot{\mathbf{v}} = \dot{\varphi}(-\sin\varphi\mathbf{t} + \cos\varphi\mathbf{t}_1) + \cos\varphi\dot{\mathbf{t}} + \sin\varphi\dot{\mathbf{t}}_1.$$

By Exercise 7.3.22, $\dot{\mathbf{t}} = \kappa_n\mathbf{N} + \kappa_g\mathbf{t}_1$, $\dot{\mathbf{t}}_1 = -\kappa_g\mathbf{t} + \tau_g\mathbf{N}$, so

$$\dot{\mathbf{v}} = (\dot{\varphi} + \kappa_g)(-\sin\varphi\mathbf{t} + \cos\varphi\mathbf{t}_1) + (\kappa_n\cos\varphi + \tau_g\sin\varphi)\mathbf{N} = (\kappa_n\cos\varphi + \tau_g\sin\varphi)\mathbf{N},$$

by Proposition 13.6.1, and hence

$$\parallel \dot{\mathbf{v}} \parallel^2 = (\kappa_n\cos\varphi + \tau_g\sin\varphi)^2.$$

13.7.1 Take the reference tangent vector field to be $\boldsymbol{\xi} = (1, 0)$, and take the simple closed curve $\boldsymbol{\gamma}(s) = (\cos s, \sin s)$. At $\boldsymbol{\gamma}(s)$, we have $\mathbf{V} = (\alpha, \beta)$, where

$$\alpha + i\beta = \begin{cases} (\cos s + i\sin s)^k & \text{if } k > 0, \\ (\cos s - i\sin s)^{-k} & \text{if } k < 0. \end{cases}$$

By de Moivre's theorem, $\alpha = \cos ks$, $\beta = \sin ks$ in both cases. Hence, the angle ψ between $\mathbf{V}$ and $\boldsymbol{\xi}$ is equal to ks, and Definition 13.7.2 shows that the multiplicity is k.

13.7.2 If $\boldsymbol{\sigma}(u, v) = \tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$, where $(\tilde{u}, \tilde{v}) \mapsto (u, v)$ is a reparametrization map, then $\mathbf{V} = \alpha\boldsymbol{\sigma}_u + \beta\boldsymbol{\sigma}_v = \tilde{\alpha}\tilde{\boldsymbol{\sigma}}_{\tilde{u}} + \tilde{\beta}\tilde{\boldsymbol{\sigma}}_{\tilde{v}} \implies \tilde{\alpha} = \alpha\frac{\partial\tilde{u}}{\partial u} + \beta\frac{\partial\tilde{v}}{\partial u}$, $\tilde{\beta} = \alpha\frac{\partial\tilde{u}}{\partial v} + \beta\frac{\partial\tilde{v}}{\partial v}$. Hence, $\tilde{\alpha}$ and $\tilde{\beta}$ are smooth if α and β are smooth. Since the components of the vectors $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are smooth, if $\mathbf{V}$ is smooth so are its components. If the components of $\mathbf{V} = \alpha\boldsymbol{\sigma}_v + \beta\boldsymbol{\sigma}_v$ are smooth, then $\mathbf{V}.\boldsymbol{\sigma}_u$ and $\mathbf{V}.\boldsymbol{\sigma}_v$ are smooth functions, hence $\alpha = \frac{G(\mathbf{V}.\boldsymbol{\sigma}_u) - F(\mathbf{V}.\boldsymbol{\sigma}_v)}{EG - F^2}$, $\beta = \frac{E(\mathbf{V}.\boldsymbol{\sigma}_v) - F(\mathbf{V}.\boldsymbol{\sigma}_u)}{EG - F^2}$ are smooth functions, so $\mathbf{V}$ is smooth.

13.7.3 If $\tilde{\psi}$ is the angle between $\mathbf{V}$ and $\tilde{\boldsymbol{\xi}}$, we have $\tilde{\psi} - \psi = \theta$ (up to multiples of 2π); so we must show that $\int_0^{\ell(\boldsymbol{\gamma})} \dot{\theta}\, ds = 0$ (a dot denotes d/ds). This is not obvious since θ is not a well defined smooth function of s (although $d\theta/ds$ is well defined). However, $\rho = \cos\theta$ is well defined and smooth, since $\rho = \boldsymbol{\xi}.\tilde{\boldsymbol{\xi}} / \parallel \boldsymbol{\xi} \parallel \parallel \tilde{\boldsymbol{\xi}} \parallel$. Now, $\dot{\rho} = -\dot{\theta}\sin\theta$, so we must prove that $\int_0^{\ell(\boldsymbol{\gamma})} \frac{\dot{\rho}}{\sqrt{1-\rho^2}} ds = 0$. Using Green's theorem, this integral is equal to

$$\int_{\boldsymbol{\pi}} \frac{\rho_u du + \rho_v dv}{\sqrt{1-\rho^2}} = \int_{\mathrm{int}(\boldsymbol{\pi})} \left(\frac{\partial}{\partial u}\left(\frac{\rho_v}{\sqrt{1-\rho^2}} \right) - \frac{\partial}{\partial v}\left(\frac{\rho_u}{\sqrt{1-\rho^2}} \right) \right) du dv,$$

where $\boldsymbol{\pi}$ is the curve in U such that $\boldsymbol{\gamma}(s) = \boldsymbol{\sigma}(\boldsymbol{\pi}(s))$; and this line integral vanishes because

$$\frac{\partial}{\partial u}\left(\frac{\rho_v}{\sqrt{1-\rho^2}} \right) = \frac{\partial}{\partial v}\left(\frac{\rho_u}{\sqrt{1-\rho^2}} \right) \quad \left(= \frac{\rho_{uv}(1-\rho^2) + \rho\rho_u\rho_v}{(1-\rho^2)^{3/2}} \right).$$

13.8.1 Let $F : \mathcal{S} \to \mathbb{R}$ be a smooth function on a surface $\mathcal{S}$, let $\mathbf{p} \in \mathcal{S}$, let $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ be patches of $\mathcal{S}$ containing $\mathbf{p}$, say $\boldsymbol{\sigma}(u_0, v_0) = \tilde{\boldsymbol{\sigma}}(\tilde{u}_0, \tilde{v}_0) = \mathbf{p}$, and let $f = F \circ \boldsymbol{\sigma}$ and $\tilde{f} = F \circ \tilde{\boldsymbol{\sigma}}$. Then, $\tilde{f}_{\tilde{u}} = f_u \frac{\partial u}{\partial \tilde{u}} + f_v \frac{\partial v}{\partial \tilde{u}}$, $\tilde{f}_{\tilde{v}} = f_u \frac{\partial u}{\partial \tilde{v}} + f_v \frac{\partial v}{\partial \tilde{v}}$, so if $f_u = f_v = 0$ at (u_0, v_0), then $\tilde{f}_{\tilde{u}} = \tilde{f}_{\tilde{v}} = 0$ at $(\tilde{u}_0, \tilde{v}_0)$.

Since $f_u = f_v = 0$ at $\mathbf{p}$, we have $\tilde{f}_{\tilde{u}\tilde{u}} = f_{uu} \left(\frac{\partial u}{\partial \tilde{u}}\right)^2 + 2f_{uv} \frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{u}} + f_{vv} \left(\frac{\partial v}{\partial \tilde{u}}\right)^2$, with similar expressions for $\tilde{f}_{\tilde{u}\tilde{v}}$ and $\tilde{f}_{\tilde{v}\tilde{v}}$. This gives, in an obvious notation, $\tilde{\mathcal{H}} = J^t \mathcal{H} J$, where $J = \begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix}$ is the Jacobian matrix of the reparametrization map $(\tilde{u}, \tilde{v}) \mapsto (u, v)$. Since J is invertible, $\tilde{\mathcal{H}}$ is invertible if $\mathcal{H}$ is invertible.

Since the matrix $\mathcal{H}$ is real and symmetric, it has eigenvectors $\mathbf{v}_1, \mathbf{v}_2$, with eigenvalues λ_1, λ_2, say, such that $\mathbf{v}_i^t \mathbf{v}_j = 1$ if $i = j$ and $= 0$ if $i \neq j$. Then, if $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$ is any vector, where α_1, α_2 are scalars, $\mathbf{v}^t \mathcal{H} \mathbf{v} = \lambda_1 \alpha_1^2 + \lambda_2 \alpha_2^2$; hence, $\mathbf{v}^t \mathcal{H} \mathbf{v} > 0$ (resp. < 0) for all $\mathbf{v} \neq \mathbf{0} \iff \lambda_1$ and λ_2 are both > 0 (resp. both < 0) $\iff \mathbf{p}$ is a local minimum (resp. local maximum); and hence $\mathbf{p}$ is a saddle point $\iff \mathbf{v}^t \mathcal{H} \mathbf{v}$ can be both > 0 and < 0, depending on the choice of $\mathbf{v}$. Since J is invertible, a vector $\tilde{\mathbf{v}} \neq \mathbf{0} \iff \mathbf{v} = J\tilde{\mathbf{v}} \neq \mathbf{0}$; and $\tilde{\mathbf{v}}^t \tilde{\mathcal{H}} \tilde{\mathbf{v}} = \tilde{\mathbf{v}}^t J^t \mathcal{H} J \tilde{\mathbf{v}} = \mathbf{v}^t \mathcal{H} \mathbf{v}$. The assertions in the last sentence of the exercise follow from this.

13.8.2 (i) $f_x = 2x - 2y$, $f_y = -2x + 8y$, so $f_x = f_y = 0$ at the origin. $f_{xx} = 2, f_{xy} = -2, f_{yy} = 8$, so $\mathcal{H} = \begin{pmatrix} 2 & -2 \\ -2 & 8 \end{pmatrix}$. $\mathcal{H}$ is invertible so the origin is non-degenerate; and the eigenvalues $5 \pm \sqrt{13}$ of $\mathcal{H}$ are both > 0, so it is a local minimum.
(ii) $f_x = f_y = 0$ and $\mathcal{H} = \begin{pmatrix} 2 & 4 \\ 4 & 0 \end{pmatrix}$ at the origin; $\det \mathcal{H} = -16 < 0$, so the eigenvalues of $\mathcal{H}$ are of opposite sign and the origin is a saddle point.
(iii) $f_x = f_y = 0$ and $\mathcal{H} = 0$ at the origin, which is therefore a degenerate critical point.

13.8.3 Using the parametrization $\boldsymbol{\sigma}$ in Exercise 4.2.5 (with $a = 2, b = 1$) gives $f(\theta, \varphi) = F(\boldsymbol{\sigma}(\theta, \varphi)) = (2 + \cos\theta) \cos\varphi + 3$. Then, $f_\theta = -\sin\theta \cos\varphi$, $f_\varphi = -(2 + \cos\theta) \sin\varphi$; since $2 + \cos\theta > 0$, $f_\varphi = 0 \Longrightarrow \varphi = 0$ or π, and then $f_\theta = 0 \Longrightarrow \theta = 0$ or π; so there are four critical points, $\mathbf{p} = (3, 0, 0)$, $\mathbf{q} = (1, 0, 0)$, $\mathbf{r} = (-1, 0, 0)$ and $\mathbf{s} = (-3, 0, 0)$. Next, $\mathcal{H} = \begin{pmatrix} -\cos\theta \cos\varphi & \sin\theta \sin\varphi \\ \sin\theta \sin\varphi & -(2 + \cos\theta) \cos\varphi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$ at $\mathbf{p}$, $= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ at $\mathbf{q}$, $= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ at $\mathbf{r}$, and $= \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ at $\mathbf{s}$; hence, $\mathbf{p}$ is a local maximum, $\mathbf{q}$ and $\mathbf{r}$ are saddle points, and $\mathbf{s}$ is a local minimum (all of which is geometrically obvious).

13.8.4 Since the torus is compact, any smooth function f on the torus must achieve its maximum and minimum values, and these are of course critical points of multiplicity 1. Since the Euler number of the torus is zero, Theorem 13.8.6 implies that f must have at least two saddle points (more if f has other local

maxima or minima).

13.8.5 The position of the two rods can be specified by giving the angles θ and φ made by the rods with the vertical. The bijection takes this configuration of the rods to the point $\boldsymbol{\sigma}(\theta, \varphi)$ of the torus, in the notation of the solution of Exercise 4.2.5.

The potential energy is $E = \frac{1}{2}a\alpha\cos\theta + (a + \frac{1}{2}b)\beta\cos\varphi +$ constant, where a, b are the lengths of the two rods and α, β are positive constants; write this as $E = \lambda\cos\theta + \mu\cos\varphi + \nu$ for simplicity (where $\lambda, \mu > 0$ and ν are constants). For a critical point we must have $\partial E/\partial\theta = \partial E/\partial\varphi = 0$, which gives $\sin\theta = \sin\varphi = 0$. Thus, there are four critical points, namely $(\theta, \varphi) = (0, 0)$, $(\pi, 0)$, $(0, \pi)$, (π, π). In the notation of Definition 13.8.3,

$$\mathcal{H} = \begin{pmatrix} -\lambda\cos\theta & 0 \\ 0 & -\lambda\cos\varphi \end{pmatrix},$$

which is obviously invertible if $\theta, \varphi = 0$ or π, so all the critical points are non-degenerate.

At $(0, 0)$ the matrix $\mathcal{H}$ has two negative eigenvalues, at $(\pi, 0)$ and $(0, \pi)$ it has one, and at (π, π) it has none. So $(0, 0)$ is a local maximum, $(\pi, 0)$ and $(0, \pi)$ are saddle points and (π, π) is a local minimum. The left-hand side of the equation in the statement of Theorem 13.8.6 is $1 - 2 + 1 = 0$, which is indeed the Euler number of the torus (Example 13.8.7).

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